## Chapter 2

## PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

**INTRODUCTION:** An equation is said to be of order two, if it involves at least one of the differential coefficients $r = (\partial^2 z / \partial^2 x)$, $s = (\partial^2 z / \partial x\, \partial y)$, $t = (\partial^2 z / \partial^2 y)$, but now of higher order; the quantities p and q may also enter into the equation. Thus the general form of a second order Partial differential equation is

$$f(x, y, z, p, q, r, s, t) = 0 \qquad ...(1)$$

The most general linear partial differential equation of order two in two independent variables x and y with variable coefficients is of the form

$$Rr + Ss + Tt + Pp + Qq + Zz = F \qquad ...(2)$$

where $R, S, T, P, Q, Z, F$ are functions of $x$ and $y$ only and not all $R, S, T$ are zero.

**Ex.1:** Solve $r = 6x$.

**Sol.** The given equation can be written as $\frac{\partial^2 z}{\partial x^2} = 6x$ ...(1)

Integrating (1) w. r. t. $x \frac{\partial z}{\partial x} = 3x^2 + \emptyset_1(y)$ ...(2)

where $\emptyset_1(y)$ is an arbitrary function of $y$.

Integrating (2) w. r. t. we get

$$x\, z = x^3 + x\, \emptyset_1(y) + \emptyset_2(y)$$

where $\emptyset_2$(y) is an arbitrary function of y.

**Ex.2.** $ar = xy$

**Sol:** Given equation can be written as $\frac{\partial^2 z}{\partial x^2} = \frac{1}{a} xy$ ...(1)

Integrating (1) w. r. t., $x$, we get

$$\frac{\partial z}{\partial x} = \left(\frac{y}{a}\right) \frac{x^2}{2} + \emptyset_1(y) \qquad ...(2)$$

where $\emptyset_1$(y) is an arbitrary function of y

Integrating (2) w. r. t., x,

$$z = \left(\frac{y}{a}\right) \frac{3}{6} + x\, \emptyset_1(y) + \emptyset_2(y)$$

or $z = \frac{y}{2a} + x\, \phi_1(y) + \phi_2(y)$

where $\phi_2(y)$ is an arbitrary function of y.

**Ex.3:** Solve $r = 2y^2$

**Sol**: Try yourself.

**Ex. 4.** Solve $t = \sin(xy)$

**Sol.** Given equation can be written as $\frac{\partial^2 z}{\partial y^2} = \sin(xy)$...(1)

Integrating (1) w. r. t., y

$$\frac{\partial z}{\partial y} = -\left(\frac{1}{x}\right)\cos(xy) + \phi_1(x) \qquad \ldots(2)$$

Integrating (2) w. r. t., y

$$z = -\left(\frac{1}{x^2}\right)\sin(xy) + y\,\phi_1(x) + \phi_2(x)$$

which is the required solution, $\phi_1$, $\phi_2$ being arbitrary functions.

**Exercises:** $xys = 1$

**Sol:** We know that $s = \frac{\partial^2 z}{\partial x \partial y}$

Therefore $xy\frac{\partial^2 z}{\partial x \partial y} = 1$

or $\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{xy}$

Integrating w.r.t., y we have

$$\frac{\partial z}{\partial x} = \frac{1}{x}\log y + f(x)$$

Again integrating w.r.t., x we get

$$z = \log x \log y + \int f(x)\, dx + F(y)$$

Or $$z = \log x \log y + g(x) + F(y)$$

**Exercises:** $2x + 2y = s$

**Sol:** The given equation can be written as

$$\frac{\partial^2 z}{\partial x \partial y} = 2x + 2y$$

Integrating w.r.t., $y$, we have

$$\frac{\partial z}{\partial x} = y^2 + 2xy + f(x)$$

Integrating w.r.t., $x$, we have

$$z = y^2 x + x^2 y + \int f(x)\,dx + F(y)$$

$$\therefore z = y^2 x + x^2 y + g(x) + F(y)$$

**Exercises:** $xr + p = 9x^2y^3$

**Sol:** The given equation can be written as

$$x\frac{\partial^2 z}{\partial x^2} + p = 9x^2y^3$$

$$\Rightarrow x\frac{\partial p}{\partial x} + p = 9x^2y^3$$

$\Rightarrow \frac{\partial p}{\partial x} + \frac{p}{x} = 9xy^3$ … (1)

which is linear first order differential equation in $p$

$\therefore$ I. F. is $e^{\log x} = x$

Multiplying (1) by $x$ we get

$$x\left[\frac{\partial p}{\partial x} + \frac{p}{x}\right] = 9x^2y^3$$

$$\Rightarrow px = 9\int x^2y^3\,dx$$

$$\Rightarrow px = 9\frac{x^3y^3}{3} + f(y)$$

$$\Rightarrow px = 3x^3y^3 + f(y)$$

$$\Rightarrow p = \frac{3x^3y^3 + f(y)}{x}$$

$$\Rightarrow \frac{\partial z}{\partial x} = 3x^2y^3 + \frac{f(y)}{x}$$

Integrating with respect to $x$ we get

$$z = x^3y^3 + f(y)\log x + F(y)$$

**Exercises:**$yt - q = xy$

**Sol:** Please try yourself.

**Exercises:**$t - xq = x^2$

**Sol:** Please try yourself.

**Exercises:**$r = 2y^2$

**Sol:** The given equation can be written as

$$\frac{\partial^2 z}{\partial x^2} = 2y^2$$

$$\Rightarrow \frac{\partial p}{\partial x} = 2x^2$$

Integrating with respect to $x$ we get

$$p = 2y^2x + f(y)$$

$$\Rightarrow \frac{\partial z}{\partial x} = 2y^2x + f(y)$$

Integrating we get

$$z = y^2x^2 + \int f(y)\,dx + F(y)$$

$$\Rightarrow z = y^2x^2 + xf(y) + F(y)$$

**Exercises:**$t = \sin(xy)$

**Sol:** Please try yourself.

**Exercises:**$\log s = x + y$

**Sol:** The given equation can be written as

$$\log \frac{\partial q}{\partial x} = x + y$$

$$\Rightarrow \frac{\partial q}{\partial x} = e^{x+y}$$

$$\Rightarrow \frac{\partial q}{\partial x} = e^x e^y$$

Integrating w.r.t. $x$ we get

$$q = e^x e^y + f(y)$$

$$\Rightarrow \frac{\partial z}{\partial y} = e^x e^y + f(y)$$

Integrating w.r.t., $y$, we get

$$z = e^x e^y + \int f(y)\, dy + F(x)$$

or $z = e^x e^y + g(y) + F(x)$

**Exercises:**$s - t = \frac{x}{y^2}$

**Sol:** Please try yourself.

**Exercises:**$t + s + q = 0$

**Sol:** The given equation can be written as

$$\frac{\partial q}{\partial y} + \frac{\partial p}{\partial y} + \frac{\partial z}{\partial y} = 0$$

Integrating with respect to $y$, we get

$$q + p + z = f(x)$$

$$\Rightarrow p + q = f(x) - z$$

It is of the form $Pp + Qq = R$

Its auxiliary system is

$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{f(x) - z}$ ...(1)

From first two fractions of (1) we get

$$dx = dy$$

$$\Rightarrow \; x - y = a$$

From first and third fractions of (1) we get

$$\frac{dx}{1} = \frac{dz}{f(x) - z}$$

$$\Rightarrow [f(x) - z]dx = dz$$

$$\Rightarrow \frac{dz}{dx} = f(x) - z$$

$$\Rightarrow \frac{dz}{dx} - z = f(x)$$

It is first order linear differential equation in $z$

Its integrating factor is $e^{\int dx} = e^{x}$

Therefore $ze^{x} = \int f(x)\, e^{x} dx$

$$\Rightarrow ze^{x} = \int f(x)\, e^{x} dx + f(y)$$

**Exercise:**$t + s + q = 1$

**Sol:** Please try yourself.

**Exercise:** Find the surface passing through the parabolas,

$$y^2 = 4ax, \qquad z = 0$$

and $y^2 = -4ax, \qquad z = 1$

and satisfying the equation $xr + 2p = 0.$

**Sol:** The given second order partial differential equation is

$$xr + 2p = 0$$

$$\Rightarrow \frac{\partial p}{\partial x} + \frac{2}{x} p = 0 \qquad \ldots (1)$$

It is first order linear differential equation in $p$.

Its integrating factor is $e^{\int \frac{2}{x} dx} = e^{2 \log x} = e^{\log x^2} = x^2$

From (1) we get

$$x^2p = \int 0\,dx$$

$$\Rightarrow x^2p = \int 0\,dx + f(y)$$

$$\Rightarrow p = \frac{f(y)}{x^2}$$

Integrating w. r. t. $x$ we have

$z = -\frac{1}{x}f(y) + F(y)$ … (2)

Using the given condition $z = 0,\ x = \frac{y^2}{4a}$, in equation (2), we have

$$0 = -\frac{4a}{y^2}f(y) + F(y)$$

or $F(y) = \frac{4af(y)}{y^2}$ … (3)

Also for $z = 1$, and $x = \frac{-y^2}{4a}$, we have from (2) we have

$$1 = \frac{4a}{y^2}f(y) + F(y)$$

Using (3) we get

or $1 = \frac{4af(y)}{y^2} + \frac{4af(y)}{y^2}$

$$\Rightarrow 1 = \frac{8af(y)}{y^2}$$

$$\Rightarrow f(y) = \frac{y^2}{8a}$$

Substituting $f(y)$, in (3)

$$F(y) = \frac{4a}{y^2}\frac{y^2}{8a}$$

$$\Rightarrow F(y) = \frac{1}{2}$$

Therefore from (1) we get

$$z = \frac{-y^2}{8ax} + \frac{1}{2}$$

Which is the required surface passing through the parabolas.

**Exercise:** Find the surface satisfying $t = 6x^3y$ containing the two lines

$y = 0 =$ zand $y = 1 = z$

**Sol:** The given 2nd order PDE is

$$t = 6x^3y$$

$$\Rightarrow \frac{\partial q}{\partial y} = 6x^3y$$

Integrating w. r. t., $y$, we have

$$q = \frac{6x^3y^2}{2} + f(x)$$

$$\Rightarrow \frac{\partial z}{\partial y} = 3x^3y^2 + f(x)$$

Integrating w. r. t., $y$,

$$z = \frac{3x^3y^3}{3} + yf(x) + F(x)$$

$\Rightarrow z = x^3y^3 + yf(x) + F(x)$ ...(1)

Using given conditions $y = 0 = z$, in (1), we have

$$0 = 0 + 0 + F(x)$$

$\Rightarrow F(x) = 0$ ... (2)

Also using $y = 1 = z$ in equation (1) we get,

$$1 = x^3 + f(x) + F(x)$$

Using (2), we get $1 = x^3 + f(x) + 0$

$f(x) = 1 - x^3$ ...(3)

Using (2) and (3) in (1) we get

$$z = x^3y^3 + y(1 - x^3)$$

Which is the required surface containing the two lines.

**Exercise:** Find the surface satisfying $r + s = 0$, and touching the elliptic paraboloid $z = 4x^2 + y^2$ along the surface of plane $y = 2x + 1$.

**Sol:** From the given equation we have $\frac{\partial p}{\partial x} + \frac{\partial q}{\partial x} = 0$.

Integrating with respect to $x$, we have

$$p + q = f(y)$$

Now, the auxiliary system is

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{f(y)} \qquad \ldots(1)$$

Taking first two fractions we get

$$\frac{dx}{1} = \frac{dy}{1}$$

Integrating we get

$$x = y + a$$

$$\Rightarrow \qquad x - y = a \qquad \ldots(2)$$

Also from 2nd and 3rd fractions of (1), we get

$$\frac{dy}{1} = \frac{dz}{f(y)}$$

$$\Rightarrow dz = f(y)dy$$

$$\Rightarrow z = \varphi(y) + b$$

or $\qquad z = \varphi(y) + F(a)$

$$\Rightarrow z = \varphi(y) + F(x - y) \qquad \ldots(3)$$

From (3), we get

$$p = \frac{\partial z}{\partial x} = F'(x - y) \qquad \ldots(4)$$

$$q = \frac{\partial z}{\partial y} = \varphi'(y) - F'(x - y) \qquad \ldots(5)$$

Since $z = 4x^2 + y^2$

$$\therefore p = \frac{\partial z}{\partial x} = 8x \qquad \ldots(6)$$

& $$q = \frac{\partial z}{\partial y} = 2y \qquad \ldots(7)$$

From (4) and (6)

$$F'(x-y) = 8x \qquad \ldots(8)$$

From (5) and (7)

$$\varphi'(y) - F'(x-y) = 2y \qquad \ldots(9)$$

Adding (8) and (9) we get

$$\varphi'(y) = 8x + 2y$$

$$= \frac{8}{2}(y-1) + 2y$$

$$= 6y - 4$$

Integrating w. r. t., $y$, we get

$$\varphi(y) = 3y^2 - 4y + b \qquad \ldots(10)$$

Also, from (8)

$$-F'(x-y) = 8x = -8(y-x-1) = 8(x-y+1)$$

Integrating w. r. t., $(x-y)$ we get

$$-F(x-y) = 4(x-y)^2 + 8(x-y) + c \qquad \ldots(11)$$

Substituting (10) and (11) in (3) we get

$$z = 3y^2 - 4y + b - 4(x-y)^2 - 8(x-y) + c$$

$$= -4x^2 - y^2 + 4y - 8x + 8xy + d$$

From the given condition,

$$4x^2 + (2x+1)^2 = -4x^2 - (2x+1)^2 + 4(2x+1) - 8x + 8x(2x+1) + d$$

$$\Rightarrow 8x^2 + 2(2x+1)^2 = 4(2x+1) - 8x + 8x(2x+1) + d$$

$$\Rightarrow 8x^2 + 8x^2 + 2 + 8x = 8x + 4 - 8x + 16x^2 + 8x + d$$

$$\Rightarrow d = -2$$

Therefore $\quad z = -4x^2 - y^2 + 4y - 8x + 8xy - 2$

which is required surface.

**Exercise:** Show that the surface satisfying $r = 6x + 2$ and touching $z = x^3 + y^3$ along its section by the plane $x + y + 1 = 0$ is $z = x^3 + y^3 + (x + y + 1)^2$.

**Sol:** Try yourself.

## Partial differential equations with constant coefficients:

We know that the general form of a linear partial differential equation

$$A_n \frac{\partial^n z}{\partial x^n} + A_{n-1}\frac{\partial^n z}{\partial x^{n-1}\partial y} + A_{n-2}\frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + A_1 \frac{\partial^n z}{\partial y^n} = f(x,y) \qquad \ldots (1)$$

Where the coefficients $A_n, \quad A_{n-1}, \quad A_{n-2}, \quad . . . , A_1$ are constants or functions of $x$ and $y$. If $A_n, \quad A_{n-1}, \quad A_{n-2}, \quad . . . , A_1$ are all constants, then (1) is called a linear partial differential equation with constant coefficients.

We denote $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ by $D\ (or\ D_x)$ and $D'(or\ D_y)$ respectively.

Therefore (1) can be written as

$$[A_n D^n + A_{n-1}D^{n-1}D' + A_{n-2}D^{n-2}D'^2 + \ \ldots\ + A_1 D'^n]z = f(x,y) \qquad \ldots (2)$$

or $\quad \varphi(D, D')z = f(x,y)$

The complementary function of (2) is given by

$$[A_n D^n + A_{n-1}D^{n-1}D' + A_{n-2}D^{n-2}D'^2 + \cdots + A_1 D'^n]z = 0 \qquad \ldots (3)$$

or $\quad \varphi(D, D')z = 0$

Let $z = F(y + mx)$ be the part of the solution

$$Dz = \frac{\partial z}{\partial x} = mF'(y + mx)$$
$$D^2 z = \frac{\partial^2 z}{\partial x^2} = m^2 F''(y + mx)$$

.. .. .. ...

.. .. .. ...

.. .. .. ...

$$D^n z = \frac{\partial^n z}{\partial x^n} = m^n F^n(y + mx)$$

And

$$D^{'} z = \frac{\partial z}{\partial y} = F^{'}(y + mx)$$

$$D^{'2} z = \frac{\partial^2 z}{\partial y^2} = F^{''}(y + mx)$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$\ldots \qquad \ldots \qquad \ldots \quad \ldots$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$D^{'n} z = \frac{\partial^n z}{\partial y^n} = F^n(y + mx)$$

Substitute these values in (3), we get

$$[A_n m^n + A_{n-1} m^{n-1} + A_{n-2} m^{n-2} + \ . \ . \ . \ + A_1] F^{(n)}(y + mx) = 0$$

which is true if $'m'$ is a root of the equation

If $m_1,\ m_2,\qquad m_n,$ are distinct roots, then complementary functions is

$$z = \varphi_1(y + m_1 x) + \varphi_2(y + m_2 x) + \ . \ . \ . \ + \varphi_n(y + m_n x)$$

where $\varphi_1, \varphi_2,\ . \ . \ .,\quad \varphi_n$ are arbitrary functions.

$$\therefore \varphi(D, D')z = 0$$

we replace $D$ by m and $D'$ by 1 to get the auxiliary equation from which we get roots.

## Linear partial differential equations with constant coefficients

**Homogenous and Non homogenous linear equations with constant coefficients:** A partial differential equation in which the dependent variable and its derivatives appear only in the first degree and are not multiplied together, their coefficients being constants or functions of x and y, is known as a linear partial differential equation. The general form of such an equation is

$$\left[A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial y \partial x^{n-1}} + A_1 \frac{\partial^n z}{\partial y^2 \partial x^{n-2}} + \ . \ . \ . \ + A_n \frac{\partial^n z}{\partial y^n}\right] + \left[B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + B_1 \frac{\partial^{n-1} z}{\partial y \partial x^{n-2}} + \right.$$
$$\left. B_1 \frac{\partial^{n-1} z}{\partial y^2 \partial x^{n-3}} + \ . \ . \ . \ + B_n \frac{\partial^{n-1} z}{\partial y^{n-1}}\right] + \left[M_0 \frac{\partial z}{\partial x} + M_1 \frac{\partial z}{\partial y}\right] + N_0 z = f(x, y) \qquad \ldots(1)$$

where the coefficients $A_0, A_1, \ldots A_n, B_0, B_1, \ldots, B_{n-1}, M_0, M_1$ and $N_0$ are all constants, then (1) is called a linear partial differential equation with constant coefficients.

For convenience $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ will be denoted by D and $D'$ respectively.

**2.1.** Short method for finding the P.I. in certain cases of $F(D,D')\, z = f(x, y)$

**2.1.a.** Short method I. when f(x, y) is of the form $\emptyset\,(ax + by)$

**Ex.1.** Solve $(D^2 + 3DD' + 2D'^2)z = x + y$

**Sol.** The Auxiliary equation of the given equation is

$m^2 + 3m + 2 = 0$ giving $m = -1, -2$

therefore C.F. = $\emptyset_1(y-x) + \emptyset_2(y-2x)$, $\emptyset_1$, $\emptyset_2$ being arbitrary functions

Now P.I. $= \frac{1}{D2 + 3DD' + 2D'2}(x + y)$

$$= \frac{1}{1^2+3.1.1+2.1^2} \iint v\,dv\,dv, \text{ where } v = x+y$$

$$= \int \left(\frac{v^2}{2}\right) dv = \frac{1}{6}\frac{v^3}{6} = \frac{1}{36}(x + y)^3$$

Hence the required general solution is z = C.F. + P.I.

or $z = \emptyset_1(y-x) + \emptyset_2(y-2x) + \frac{1}{36}(x + y)^3$

**Ex. 2.** Solve $(2D^2 - 5DD' + 2D'^2)\, z = 24\,(y - x)$

**Sol:** Try yourself

**Ex.3.** Solve $(D^2 + 3DD' + 2D'^2)z = 2x + 3y$

**Sol:** Try yourself

**2.1.b.Short method II.**

When f(x,y) is of the form $x^m y^n$ or a rational integral

**Ex.1.** Solve $(D^2 - a^2D'^2)z = x$

**Sol.** Here auxiliary equation is $m^2 - a^2 = 0$ so that $m = a, -a$

Therefore C.F. $= \emptyset_1(y + ax) + \emptyset_2(y - ax)$, ...(1)

$\emptyset_1, \ \emptyset_2$ being arbitrary functions.

Now P.I. = $\frac{1}{D^2 - a^2 D'^2}(x) = \frac{1}{D^2\left[1 - a^2\left(\frac{D'^2}{D^2}\right)\right]}(x)$

$= \frac{1}{D2}\left[1 - a^2\left(\frac{D'^2}{D^2}\right)\right]^{-1}(x)$

$= \frac{1}{D2}[1 + a^2 (D'^2/D^2) + \ldots]x$

$= \frac{1}{D2}x$

$= \frac{x^3}{6}$ ...(2)

Hence the required solution is z = C.F. + P.I.

$$Z = \emptyset_1(y + a x) + \emptyset_2(y - a x) + \frac{x^3}{6}$$

**Exercise: 1.**$2r + 5s + 2t = 0$

**Sol:** It is a second order pole with constant coefficients, we have

$$2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$$

or $(2D^2 + 5DD' + 2D'^2)z = 0$

Now the auxiliary equations is given by

$$(2m^2 + 5m + 2) = 0$$

$\Rightarrow m = -\frac{1}{2}$ , -2

Therefore the complementary function is $z = \varphi_1\left(y - \frac{1}{2}x\right) + \varphi_2(y - 2x)$

which is required solution.

**Exercise: 2.**$r = a^2 t$

**Sol:** Try Yourself (Ans: $z = \varphi_1(y + ax) + \varphi_2(y - ax)$ )

**Exercise: 3.**$\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 2\frac{\partial^3 z}{\partial x \partial^2 y} = 0$

**Sol:** It is a third order pole with constant coefficients, we have

$$\frac{\partial^3 z}{\partial x^3}-3\frac{\partial^3 z}{\partial x^2\partial y}+2\frac{\partial^3 z}{\partial x\partial^2 y}=0$$

or $$(D^3-3D^2D'+2DD'^2)z=0$$

Now the auxiliary equations is given by

$$m^3-3m^2+2m=0$$

$$\Rightarrow m(m^2-3m+2)=0$$

$$\Rightarrow m(m-1)(m-2)=0$$

$\Rightarrow m=0,\ 1\ ,\ 2$

Therefore the complementary function is $z=\varphi_1(y)+\varphi_2(y+x)+\varphi_3(y+2x)$

Which is required solution.

**Exercise: 4.** $\frac{\partial^3 z}{\partial x^3}-6\frac{\partial^3 z}{\partial x^2\partial y}+11\frac{\partial^3 z}{\partial x\partial^2 y}-6\frac{\partial^3 z}{\partial y^3}=0$

**Sol:** Try yourself

**Exercise: 5.** $25r-40s+16t=0$

**Sol:** Try yourself

**Exercise: 6.** $(D^4-D'^4)z=0$

**Sol:** The auxiliary equation is given by

$$m^4-1=0$$

$$\Rightarrow (m^2-1)(m^2+1)=0$$

$$\Rightarrow (m-1)(m+1)(m-i)(m+i)=0$$

$$\Rightarrow m=1,\quad -1,\ i,\quad -i$$

Therefore the complementary function is

$$z=\varphi_1(y+x)+\varphi_2(y-x)+\varphi_3(y+ix)+\varphi_4(y-ix)$$

which is required solution.

**Exercise: 7.** $(D^3-4D^2D'+4DD'^2)z=0$

**Sol:** Try yourself.

**Exercise: 8.** $(D^2-2DD'+D'^2)z=12xy$

**Sol:** The auxiliary equations corresponding to these linear system of equations is given by

$$m^2-2m+1=0$$

$$\Rightarrow (m-1)(m-1)=0$$

$$\Rightarrow m=1, \qquad 1$$

Therefore the complementary function is

$$z=f_1(y+x)+xf_2(y+x)$$

Also, P. I. $=\frac{1}{(D-D')^2}12xy$

$$=\frac{12}{D^2}\left(1-\frac{D'}{D}\right)^{-2}xy$$

$$=\frac{12}{D^2}\left[1-2\frac{D}{D'}+\left(\frac{D}{D'}\right)^2\right]^{-1}xy$$

$$=\frac{12}{D^2}\left[1-\left\{-2\frac{D}{D'}+\left(\frac{D}{D'}\right)^2\right\}+\quad\right]xy$$

$$=\frac{12}{D^2}\left(xy+\frac{2}{D}x\right)$$

$$=\frac{12}{D}\left(\frac{x^2y}{2}+\frac{x^3}{3}\right)$$

$$=12\left(\frac{x^3y}{6}+\frac{x^4}{12}\right)$$

$$=2x^3y+x^4$$

Therefore the complete solution is z=C. F. + P. I.

i.e., $z=f_1(y+x)+xf_2(y+x)+2x^3y+x^4$

**Exercise: 9.** $(2D^2-5DD'+2D'^2)z=24(x-y)$

**Sol:** Try yourself.

**Exercise: 10.** $(D^3-D'^3)z=x^3y^3$

**Sol:** The auxiliary equations is

$$m^3 - 1 = 0$$

$$\Rightarrow (m-1)(m^2+m+1) = 0$$

$$\Rightarrow m = 1,\ \frac{-1+i\sqrt{3}}{2},\ \frac{-1-i\sqrt{3}}{2}$$

Complementary function is $z = \varphi_1(y+x) + \varphi_2\left(y + \frac{-1+i\sqrt{3}}{2}x\right) + \varphi_3\left(y + \frac{-1-i\sqrt{3}}{2}x\right)$

$$\text{P. I.} = \frac{1}{D^3 - D'^3}x^3y^3$$

$$= \frac{1}{D^3\left[1-\left(\frac{D'}{D}\right)^3\right]}x^3y^3$$

$$= \frac{1}{D^3}\left[1-\left(\frac{D'}{D}\right)^3\right]^{-1}x^3y^3$$

$$= \frac{1}{D^3}\left[1+\left(\frac{D'}{D}\right)^3+\left(\frac{D'}{D}\right)^6+\ .\ .\ .\ \right]x^3y^3$$

$$= \frac{1}{D^3}\left[x^3y^3+\left(\frac{D'}{D}\right)^3x^3y^3+\left(\frac{D'}{D}\right)^6x^3y^3+\ .\ .\ .\ \right]$$

$$= \frac{1}{D^3}\left[x^3y^3+\frac{D'^2}{D^3}3x^3y^2+0+\ .\ .\ .\ \right]$$

$$= \frac{1}{D^3}\left[x^3y^3+\frac{D'}{D^3}6x^3y\right]$$

$$= \frac{1}{D^3}\left[x^3y^3+\frac{1}{D^3}6x^3\right]$$

$$= \frac{1}{D^3}\left[x^3y^3+\frac{1}{D^2}6\frac{x^4}{4}\right]$$

$$= \frac{1}{D^3}\left[x^3y^3+\frac{1}{D}6\frac{x^5}{20}\right]$$

$$= \frac{1}{D^3}\left[x^3y^3+\frac{x^6}{20}\right]$$

$$= \frac{1}{D^2}\left[\frac{x^4y^3}{4}+\frac{x^7}{140}\right]$$

$$= \frac{1}{D}\left[\frac{x^5y^3}{20}+\frac{x^8}{1120}\right]$$

$$= \left[\frac{x^6y^3}{120}+\frac{x^9}{10080}\right]$$

The complete solution is C.F. + P.I.

$$z=\varphi_1(y+x)+\varphi_2\left(y+\frac{-1+i\sqrt{3}}{2}x\right)+\varphi_3\left(y+\frac{-1-i\sqrt{3}}{2}x\right)+\frac{x^6y^3}{120}+\frac{x^9}{10080}$$

**Exercise:** Find the real function $'v'$ of $x$ and $y$ reducing to zero when $y=0$ and satisfying $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=-4\pi(x^2+y^2)$

**Sol:** We have to find the P. I. only

P. I. $= \frac{1}{D^2+D'^2}[-4\pi(x^2+y^2)]$

$$= \frac{1}{D^2\left(1+\frac{D'^2}{D^2}\right)}[-4\pi(x^2+y^2)]$$

$$= \frac{-4\pi}{D^2}\left[1+\frac{D'^2}{D^2}\right]^{-1}(x^2+y^2)$$

$$= \frac{-4\pi}{D^2}\left[1-\frac{D'^2}{D^2}+\left(\frac{D'^2}{D^2}\right)^2+\ .\ .\ .\ \right](x^2+y^2)$$

$$= \frac{-4\pi}{D^2}\left[(x^2+y^2)-\frac{D'^2}{D^2}(x^2+y^2)+0\ \right]$$

$$= \frac{-4\pi}{D^2}\left[(x^2+y^2)-\frac{D'}{D^2}(2y)\right]$$

$$= \frac{-4\pi}{D^2}\left[(x^2+y^2)-\frac{1}{D^2}(2)\right]$$

$$= \frac{-4\pi}{D^2}\left[(x^2+y^2)-\frac{1}{D}(2x)\right]$$

$$= \frac{-4\pi}{D^2}[(x^2+y^2)-(x^2)]$$

$$= \frac{-4\pi}{D}[xy^2]$$

$$= -4\pi\left[\frac{x^2y^2}{2}\right]$$

$$= -2\pi x^2y^2$$

**Theorem:** If $u_1,\ u_2,\ u_3,\ .\ .\ .\ ,u_n$ are the solutions of the homogeneous linear PDE $F(D,D')z=0$, then $\sum_{r=1}^{n} c_r u_r$ where $c_r's$ are arbitrary constants, is also a solution.

**Proof:** Since $u_r$, r=1, 2, 3, . . . , n are solutions of the PDE $F(D,D')z = 0$

So, we have $u_r$ is one of the solutions i.e.,

$F(D,D')u_r = 0$ , $r = 1,\ 2,\ .\ .\ .\ ,\ n$

$$\therefore F(D,D^{'})(c_r u_r) = c_r{}^{'} F(D,D^{'})(u_r)$$

and$F(D,D^{'})\left(\sum u_r\right) = \sum F(D,D^{'})(u_r)$

$\therefore$ for any set of functions $u_r$, we have

$$F(D,D^{'})\left(\sum_{r=1}^{n} c_r u_r\right) = \sum_{r=1}^{n} F(D,D^{'}) c_r u_r$$

$$= \sum_{r=1}^{n} c_r F(D,D^{'}) u_r$$

$$= 0$$

Therefore $\sum_{r=1}^{n} c_r u_r$ acts as a solution for the homogeneous system.

## Reducible and irreducible:

If an operator $F(D,D')$ can be expressed as a product of linear factors, it is said to be reducible. If it can not be factorised, then it is said to be irreducible.

**Theorem:** If $\alpha_r D + \beta_r D^{'} + \gamma_r$ is a factor of $F(D,D')$ and $\varphi_r(\xi)$, then

$u_r = exp.\left(-\frac{\gamma_r x}{\alpha_r}\right)\varphi_r(\beta_r x - \alpha_r y)$ for $\alpha_r$ is a solution of the equation $F(D,D')z = 0$.

**Proof:** The given equation is $F(D,D')z = 0$ . . . (1)

In order to prove $u_r = exp.\left(-\frac{\gamma_r x}{\alpha_r}\right)\varphi_r(\beta_r x - \alpha_r y);$ $\alpha_r \neq 0$ . . . (2)

is a solution of (1), we have to prove $F(D,D')u_r = 0$

Diff. eq.(2) w.r.t. $x$ and $y$, we get

$$Du_r = -\frac{\gamma_r}{\alpha_r}u_r + \beta_r exp.\left(-\frac{\gamma_r x}{\alpha_r}\right)\varphi'(\beta_r x - \alpha_r y)$$

And

$$D'u_r = -\alpha_r exp.\left(-\frac{\gamma_r x}{\alpha_r}\right)\varphi'(\beta_r x - \alpha_r y)$$

$$\therefore (\alpha_r D + \beta_r D' + \gamma_r)u_r = -\gamma_r u_r + \alpha_r\beta_r exp.\left(-\frac{\gamma_r x}{\alpha_r}\right)\varphi'(\beta_r x - \alpha_r y)$$

$-\alpha_r\beta_r\ exp.\left(-\frac{\gamma_r x}{\alpha_r}\right)\varphi'(\beta_r x - \alpha_r y) + \gamma_r u_r = 0\ .\ .\ .$ (3)

Since $(\alpha_r D + \beta_r D' + \gamma_r)$ is a factor of $F(D, D')$

Therefore $F(D, D')z = g(D, D')(\alpha_r D + \beta_r D' + \gamma_r)z,$ using (3), we get

$$F(D, D')u_r = 0$$

Therefore $u_r$ is a solution of $F(D, D')z = 0$

**Solution of Reducible Equations:**

Let $F(D, D')z = f(x, y)\ \ .\ .\ .$ (1)

be a partial differential equation. Since (1) is reducible therefore

$$F(D, D')z = \prod_{r=1}^{n}(\alpha_r D + \beta_r D' + \gamma_r)z$$

If $z$ satisfies $(\alpha_r D + \beta_r D' + \gamma_r)z = 0, r = 0, 1, 2, \ .\ .\ .\ ., n,$ then it gives us complementary function

Now $\alpha_r \frac{\partial z}{\partial x} + \beta_r \frac{\partial z}{\partial y} + \gamma_r z = 0$

The subsidiary system is

$$\frac{dx}{\alpha_r} = \frac{dy}{\beta_r} = \frac{dz}{\gamma_r z}$$

From the first two members

$$\beta_r x - \alpha_r y = c_r$$

From first and last members we get

$$\frac{dz}{z} = -\frac{\gamma_r}{\alpha_r}dx$$

Integrating we get $\log z = -\frac{\gamma_r}{\alpha_r}x + A_r$

$$\Rightarrow z = \log \beta_r \exp\left(-\frac{\gamma_r}{\alpha_r}x\right) [e^{A_r} = \beta_r]$$

$$\Rightarrow z = \varphi(r) \exp\left(-\frac{\gamma_r}{\alpha_r}x\right)$$

$$\Rightarrow z = \varphi(\beta_r x - \alpha_r y) \exp\left(-\frac{\gamma_r}{\alpha_r}x\right)$$

Also

$$\frac{dz}{z} = -\frac{\gamma_r}{\beta_r}dy$$

$$\Rightarrow z = \varphi(\beta_r x) \exp\left(-\frac{\gamma_r}{\alpha_r}y\right)$$

**Example:** Let $(\alpha_r D + \beta_r D' + \gamma_r)z_1 = 0$ where $z_1 = (\alpha_r D + \beta_r D' + \gamma_r)z$

$$z_1 = \varphi(\beta_r x - \alpha_r y) \exp\left(-\frac{\gamma_r}{\alpha_r}x\right)$$

$$\Rightarrow (\alpha_r D + \beta_r D' + \gamma_r)z = \varphi_r(\beta_r x - \alpha_r y) \exp\left(-\frac{\gamma_r}{\alpha_r}x\right)$$

$$\Rightarrow \alpha_r \frac{\partial z}{\partial x} + \beta_r \frac{\partial z}{\partial y} = \varphi_r(\beta_r x - \alpha_r y) \exp\left(-\frac{\gamma_r}{\alpha_r}x\right) - \gamma_r z$$

Auxiliary system is

$$\frac{dx}{\alpha_r} = \frac{dy}{\beta_r} = \frac{dz}{\varphi_r(\beta_r x - \alpha_r y) \exp\left(-\frac{\gamma_r}{\alpha_r}x\right) - \gamma_r z}$$

From first two we get

$$\frac{dx}{\alpha_r} = \frac{dy}{\beta_r}$$

$$\Rightarrow \beta_r dx = \alpha_r dy$$

$$\Rightarrow \beta_r x - \alpha_r dy = c_r$$

From first and third we get

$$\frac{dx}{\alpha_r} = \frac{dz}{\varphi_r(\beta_r x - \alpha_r y)\exp\left(-\frac{\gamma_r}{\alpha_r}x\right) - \gamma_r z}$$

$$\Rightarrow \frac{dz}{dx} = \frac{\varphi_r(\beta_r x - \alpha_r y)\exp\left(-\frac{\gamma_r}{\alpha_r}x\right) - \gamma_r z}{\alpha_r}$$

$$\Rightarrow \frac{dz}{dx} + \frac{\gamma_r z}{\alpha_r} = \frac{\varphi_r(\beta_r x - \alpha_r y)\exp\left(-\frac{\gamma_r}{\alpha_r}x\right)}{\alpha_r}$$

Here I. F. is $e^{\frac{\gamma_r x}{\alpha_r}}$ therefore the above equation can be written as

$$d\left(ze^{\frac{\gamma_r x}{\alpha_r}}\right) = \frac{\varphi_r(\beta_r x - \alpha_r y)}{\alpha_r}$$

$$\Rightarrow ze^{\frac{\gamma_r x}{\alpha_r}} = \frac{1}{\alpha_r}\int \varphi_r(\beta_r x - \alpha_r y)dx + \beta_r$$

$$\Rightarrow ze^{\frac{\gamma_r x}{\alpha_r}} = e^{-\frac{\gamma_r x}{\alpha_r}}[\varphi_r(\beta_r x - \alpha_r y) + \psi_r(\beta_r x - \alpha_r y)]$$

**Example:** If $z = e^{ax+by}$

Then $F(D, D')z = F(a, b)e^{ax+by}$

$z$ acts as the solution of $F(D, D')z$, where $F(D, D')z$ is reducible if $F(a, b) = 0$.

**Exercise:** $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} - \frac{\partial^3 z}{\partial x \partial y^2} + 2\frac{\partial^3 z}{\partial y^3} = e^{x+y}$

**Sol:** The given differential equations can be written as

$$(D^3 - 2D^2D' - DD'^2 + 2D'^3)z = e^{x+y}$$

Auxiliary equations $m^3 - 2m^2 - m + 2 = 0$

$$\Rightarrow (m-1)(m^2 - m - 2) = 0$$

$$\Rightarrow m = 1, \quad -1, \quad 2$$

Therefore the C. F. is

$$z = f_1(y+x) + f_1(y-x) + f_1(y+2x)$$

P. I. $= \frac{1}{D^3 - 2D^2D' - DD'^2 + 2D'^3}e^{x+y}$

$$= \frac{1}{D^2(D-2D')-D'^2(D-D')}e^{x+y}$$

$$= \frac{1}{\left(D^2-D'^2\right)(D-2D')}e^{x+y}$$

$$= \frac{1}{(D-D')(D+D')(D-2D')}e^{x+y}$$

$$= \frac{1}{(D-D')(1+1)(1-2)}e^{x+y}$$

$$= \frac{1}{-2(D-D')}e^{x+y}$$

Now let $w = \frac{1}{D-D'}e^{x+y}$

$$\Rightarrow (D-D')w = e^{x+y}$$

The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dw}{e^{x+y}}$$

From first two members we have

$$\frac{dx}{1} = \frac{dy}{-1}$$

$$\Rightarrow dx + dy = 0$$

$$\Rightarrow x + y = c$$

From first and third member we get

$$\frac{dx}{1} = \frac{dw}{e^{x+y}}$$

$$\Rightarrow dx = \frac{dw}{e^c}$$

$$\Rightarrow dw = e^c dx$$

$$\Rightarrow w = e^c x$$

$$\Rightarrow w = xe^{x+y}$$

Therefore the particular integral = $-\frac{w}{2} = -\frac{1}{2}xe^{x+y}$

Hence the complete solution is $z = C.F + P.I$

$$z = f_1(y+x) + f_1(y-x) + f_1(y+2x) + -\frac{1}{2}xe^{x+y}$$

**Laplace Equation**

$$\nabla^2 z = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) z = 0$$

**Exercise:** Find the solution of the equation $\nabla^2 \mathrm{z} = \mathrm{e}^{-\mathrm{x}} \cos \mathrm{y}$

Which tends to 0 as $x \to \infty$ and $\cos y$ for $x = 0$.

**Sol:** The given pde is $\nabla^2 \mathrm{z} = \mathrm{e}^{-\mathrm{x}} \cos \mathrm{y}$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \mathrm{e}^{-\mathrm{x}} \cos \mathrm{y}$$

$\Rightarrow (D^2 + D'^2)z = \mathrm{e}^{-\mathrm{x}} \cos \mathrm{y}$ . . .. (1)

On comparing with $F(D, D') = D^2 + D'^2$ and $f(x,y) = \mathrm{e}^{-\mathrm{x}} \cos \mathrm{y}$

Let $z = e^{ax+by}$ be the solution of (1)

$$\therefore (D^2 + D'^2)z = a^2 e^{ax+by} + b^2 e^{ax+by}$$

$$= (a^2 + b^2)e^{ax+by}$$

Where $a^2 + b^2 = 0 = F(a,b)$

Therefore the complementary function is $C.F. = \sum_{r=0}^{\infty} A_r\, e^{ax+by}$

$A_r's$ being the constants and $a_r^2 + b_r^2 = 0$

Also P. I. $= \frac{1}{D^2+D'^2} \mathrm{e}^{-\mathrm{x}} \cos \mathrm{y}$

$= \cos y \frac{1}{D^2-1} \mathrm{e}^{-\mathrm{x}}$

$= x \cos y \frac{1}{2D} \mathrm{e}^{-\mathrm{x}}$

$= -\frac{x}{2} \cos y\, \mathrm{e}^{-\mathrm{x}}$

Therefore the complete solution is

$$z = \sum_{r=0}^{\infty} A_r\, e^{ax+by} - \frac{x}{2} \cos y\, \mathrm{e}^{-\mathrm{x}}$$

Using $z \to 0$ as $x \to \infty$, we write

$a_r = -\lambda_r$ where $\lambda_r > 0$

Since $a_r^2 + b_r^2 = 0$

$$\Rightarrow b_r = \pm i\sqrt{a_r^2}$$

$$= \pm i\lambda_r$$

$$z = \sum_{r=0}^{\infty} A_r\, e^{-\lambda_r x} e^{\pm i\lambda_r x} - \frac{x}{2}\cos y\, \mathrm{e}^{-\mathrm{x}}$$

$$= \sum_{r=0}^{\infty} B_r\, e^{-\lambda_r x} \cos(\lambda_r y + \epsilon_r) - \frac{x}{2}\cos y\, \mathrm{e}^{-\mathrm{x}}$$

Using the boundary condition

$$\cos y = \sum_{r=0}^{\infty} B_r \cos(\lambda_r y + \epsilon_r)$$

Where $B_r = 1\ and\ \lambda_r = 1\quad for\, r = 0$ and $B_r = 0\ and\ \lambda_r = 0\quad for\, r \neq 0$

Therefore $z = \cos y\, \mathrm{e}^{-\mathrm{x}} - \frac{x}{2}\cos y\, \mathrm{e}^{-\mathrm{x}}$ is the required solution.

**Exercise:** Show that the equation $\frac{\partial^2 y}{\partial t^2} + 2k\frac{\partial y}{\partial t} = c^2\frac{\partial^2 y}{\partial x^2}$

Possesses solution of the form $\sum C_r\, e^{-kt}\cos(\alpha_r x + \epsilon_r)\cos(w_r t + \delta_r)$

Where $C_r,\ \alpha_r,\ \epsilon_r,\ \omega_r,\ \delta_r$ are constants and $\omega_r = \delta_r^2 c^2 - k^2$

**Sol:** Let $\frac{\partial y}{\partial t} = D;\quad \frac{\partial y}{\partial x} = D';$

Therefore $(D^2 + 2kD)y = c^2 D'^2 y$

or $(D^2 + 2kD - c^2 D'^2)y = 0$ ... (1)

It is irreducible, therefore let $y = e^{at+bx}$

$\Rightarrow\quad Dy = ae^{at+bx}$ and $D^2 y = a^2 e^{at+bx}$

Similarly we get

$D'y = be^{at+bx}$ and $D'^2 y = b^2 e^{at+bx}$

Therefore form (1) we have

$$a^2e^{at+bx}+2kae^{at+bx}+c^2b^2e^{at+bx}=0$$

$$\Rightarrow\ (a^2+2ka+c^2b^2)e^{at+bx}=0$$

$$\Rightarrow\ a^2+2ka+c^2b^2=0$$

$$\Rightarrow\ a=\frac{-2k\pm\sqrt{4k^2+4b^2c^2}}{2}$$

$$\Rightarrow\ a=-k\pm\sqrt{k^2+b^2c^2}$$

In general $a_r=-k\pm\sqrt{k^2+b_r^2c^2}$

If $b_r^2=-\alpha_r^2$

Then $a_r=-k\pm\sqrt{k^2-\alpha_r^2c^2}$

$=-k\pm i\omega_r$

Where $\omega_r^2=\alpha_r^2c^2-k^2$

Therefore $y=\ e^{b_rx}e^{a_rt}$

$=\ e^{-xt}e^{-i\omega_rt}e^{-ia_rx}$

$$y=\sum_{r=0}^{\infty}c_re^{-xt}\cos(\alpha_rx+\epsilon_r)\cos(\omega_rt+\delta_r)$$

Where $C_r,\ \alpha_r,\ \epsilon_r,\ \omega_r,\ \delta_r$ are constants and $\omega_r=\delta_r^2c^2-k^2$

**Exercise:1** If $z=f(x^2-y)+g(x^2+y)$, where $f$ and $g$ are arbitrary constants, prove that

$$\frac{\partial^2z}{\partial x^2}-\frac{1}{x}\frac{\partial z}{\partial x}=4x^2\frac{\partial^2z}{\partial y^2}$$

**Exercise:2** Find the solution of

$$\frac{\partial^2z}{\partial x^2}-\frac{\partial^2z}{\partial y^2}=x-y$$

**Sol:** The given equation is

$$(D^2 - D'^2)z = x - y$$

The auxiliary equation is $m^2 - 1 = 0$ so that $m = \pm 1$

Therefore C. F. is $f_1(y + x) + f_2(y - x)$

P. I. is $\frac{1}{D^2 - D'^2}(x - y)$

$$= \frac{1}{D^2\left(1 - \frac{D'^2}{D^2}\right)}(x - y)$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{D}{D'}\right)^2\right]^{-1}(x - y)$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{D}{D'}\right)^2 + \left(\frac{D}{D'}\right)^4 + \ ...\ \right](x - y)$$

$$= \frac{1}{D^2}\left[(x - y) + \left(\frac{D}{D'}\right)^2(x - y) + \left(\frac{D}{D'}\right)^4(x - y) + \ ...\ \right]$$

$$= \frac{1}{D^2}[(x - y) + 0 + 0\ \ ...\ ]$$

$$= \frac{1}{D}\left[\frac{x^2}{2} - yx\right]$$

$$= \frac{x^3}{3} - \frac{yx^2}{2}$$

Therefore the complete solution is $z = C.F. + P.\ I. = f_1(y + x) + f_2(y - x) + \frac{x^3}{3} - \frac{yx^2}{2}$

**Exercise:3** Find the solution of

$$\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} = 2\frac{\partial^4 z}{\partial x^2 \partial y^2}$$

**Sol:** Please try Yourself

**Exercise:4** Show that the equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{k}\frac{\partial z}{\partial t}$$

Possesses solution of the form

$$\sum_{r=0}^{\infty} c_n \cos(nx + \epsilon_n)\, e^{-kn^2 t}$$

**Sol:** Please try Yourself

**Exercise:5** Find the P. I. of the following PDE's

(a) $(D^2 - D')z = 2y - x^2$

(b) $(D^2 - D')z = e^{2x-y}$

(c) $r + s - 2t = e^{x+y}$

(d) $r - s + 2q - z = x^2y^2$

(e) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y \cos x$

**Sol:** (a) P. I. is $\frac{1}{D^2 - D'}(2y - x^2)$

$$= \frac{1}{D^2\left[1 - \frac{D'}{D^2}\right]}(2y - x^2)$$

$$= \frac{1}{D^2}\left[1 - \frac{D'}{D^2}\right]^{-1}(2y - x^2)$$

$$= \frac{1}{D^2}\left[1 + \frac{D'}{D^2} + \frac{D'^2}{D^4} + \;.\;.\;.\;\right](2y - x^2)$$

$$= \frac{1}{D^2}\left[(2y - x^2) + \frac{D'}{D^2}(2y - x^2) + \frac{D'^2}{D^4}(2y - x^2) + \;.\;.\;.\;\right]$$

$$= \frac{1}{D^2}\left[(2y - x^2) + \frac{1}{D^2}(2)\right]$$

$$= \frac{1}{D^2}\left[(2y - x^2) + \frac{1}{D}(2x)\right]$$

$$= \frac{1}{D^2}[(2y - x^2) + x^2]$$

$$= \frac{1}{D^2}[2y\,]$$

$$= \frac{1}{D}[2xy\,]$$

$$= x^2y$$

Which is required particular intergral

(b) Please try yourself.

(c) Please try yourself.

(d) P. I. is $\frac{1}{D^2-DD'+2D'-1}(x^2y^2)$

$$= \frac{1}{D^2\left[1-\frac{D'}{D}+\frac{2D'}{D^2}-\frac{1}{D^2}\right]}(x^2y^2)$$

$$= \frac{1}{D^2}\left[1-\left(\frac{D'}{D}-\frac{2D'}{D^2}+\frac{1}{D^2}\right)\right]^{-1}(x^2y^2)$$

$$= \frac{1}{D^2}\left[1+\left(\frac{D'}{D}-\frac{2D'}{D^2}+\frac{1}{D^2}\right)+\left(\frac{D'}{D}-\frac{2D'}{D^2}+\frac{1}{D^2}\right)^2+\ .\ .\ .\ \right](x^2y^2)$$

$$= \frac{1}{D^2}\Big[(x^2y^2)+\frac{1}{D}(2x^2y)-\frac{2}{D^2}(2x^2y)+\frac{1}{D^2}(x^2y^2)+\frac{D'}{D^2}(x^2y^2)+ 4\frac{D'^2}{D^4}(x^2y^2)+\frac{1}{D^4}(x^2y^2)-4\frac{D'^2}{D^3}(x^2y^2)-\frac{4}{D^4}(x^2y^2)+\frac{2D'}{D^3}(x^2y^2)+\ .\ .\ .\ \Big]$$

$$= \frac{1}{D^2}\Big[(x^2y^2)+\frac{2}{3}(x^3y)-\frac{1}{12}(x^4y)+\frac{1}{6}(x^4)+\frac{1}{40}(x^6)-\frac{2}{15}(x^5)- \frac{1}{45}(x^6y)+\frac{1}{15}(x^5y)\Big]$$

$$= \frac{x^8}{2240}-\frac{x^7}{315}+\frac{x^6}{180}+\frac{x^5y}{30}+\frac{x^4y^2}{12}-\frac{x^6y}{120}-\frac{x^8y}{2570}-\frac{x^7y}{630}$$

which is the required particular integral.

# Chapter-3

## Classification of second order partial differential equation

**Definition:** A second order partial differential equation which is linear w. r. t., the second order partial derivatives i.e. $r, s$ and $t$ is said to be a quasi linear PDE of second order. For example the equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad \dots.(1)$$

where $f(x, y, z, p, q)$ need not be linear, is a quasi linear partial differential equation. Here the coefficients R, S, T may be functions of x and y, however for the sake of simplicity we assume them to be constants.

The equation (1) is said to be

(i) Elliptic if $S^2 - 4RT < 0$
(ii) Parabolic if $S^2 - 4RT = 0$
(iii) Hyperbolic if $S^2 - 4RT > 0$

**BOUNDARY VALUE PROBLEMS:** The function v in addition to satisfying the Laplace and Poisson equations in bounded region R in three dimensional space, should also satisfy certain boundary conditions on the boundary C of this region. Such problems are referred to as Boundary value problems for Laplace and poison equations. If a function $f \in C^n$, then all its derivatives of order n are continuous. If $f \in C^0$,then we mean that f is continuous.

There are mainly three types of boundary value problems for Laplace equation. If $f \in C^{0,}$ and is prescribed on the boundary C of some finite region R, the problem of determining a function $\emptyset\,(x, y, z)$ such that$\nabla^2\emptyset = 0$ within R and satisfying $\emptyset = f$ on C,is called the boundary value problem of first kind or Dirichlet problem. The second type of boundary value problem (BVP) is to determine the function $\emptyset\,(x, y, z)$ so that $\nabla^2\emptyset = 0$ within R while $\frac{\partial\emptyset}{\partial n}$ is sepecified at every point of C, where $\frac{\partial\emptyset}{\partial n}$is the normal derivative of $\emptyset$. This problem is called the Neumann problem.

The third type of boundary value problem is concerned with the determination of the function $\emptyset\,(x, y, z)$ such that $\nabla^2\emptyset = 0$ within R,while a boundary condition of the form $\frac{\partial\emptyset}{\partial n}$+ h $\emptyset = f$,where $h \geq 0$ is specified at every point of the boundary C.This is called a mixed boundary value problem or Churchill's problem.

**Separation of variables method:**

The method of separation of variables is applicable to a large number of classical linear homogenous Equations. The choice of the coordinate system in general depends on the shape of the body. Consider a two dimensional Laplace equation

$$\nabla^2\emptyset = \frac{\partial^2(u)}{\partial x^2} + \frac{\partial^2(u)}{\partial y^2} = 0 \qquad \ldots(1)$$

We assume that $u(x, y) = X(x)\, Y(y)$ ...(2)

Equation (1) and (2) provide us

$\frac{X''}{X} = \frac{Y''}{Y} = k$ (separation parameter)

**Three cases arise :**

**Case I:** Let k>0.then $k = p^2$, p is real we get

$\frac{d^2X}{dx^2} - p^2X = 0$ and $\frac{d^2Y}{dy^2} - p^2X = 0$

which imply that $X = C_1 e^{px} + C_2 e^{-px}$

and $Y = C_3 \cos py + C_4 \sin py$

then solution is

$u(x,y) = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py)$ ...(3)

**Case II:** let k = 0 then $\frac{d^2X}{dx^2} = 0$ and $\frac{d^2Y}{dy} = 0$

Which provide us $X = C_5x + C_6$ and $Y = C_7y + C_8$

The solution is therefore $u(x,y) = (C_5x + C_6)(C_7y + C_8)$ ...(4)

**CASE III:** let k < 0 then $k = -p^2$ proceeding as in case I, we obtain

$u(x,y) = (C_9 \cos px + C_{10} \sin px)(C_{11} e^{py} + C_{12} e^{-py})$ ...(5)

In all these cases $C_i$ $(i = 1, 2, 3, \ldots, 12)$ are integration constants,which are calculated by using the boundary conditions. For example, consider the boundary condition

$U(x,0) = 0$, $u(x, a) = 0$, $u(x, y) \to 0$, as $x \to \infty$

Where $x \geq 0$ and $0 \leq y \leq a$.

The appropriate solution for u(x, y) by the methods of separation of variables obtained above in this case is

$$U(x, y) = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py) \qquad ...(6)$$

Since $u(x,y) \to 0$, as $x \to \infty$, we have

$C_1 = 0 \ \forall \ y$

$(x,y) = C_2 e^{-px} (C_3 \cos py + C_4 \sin py)$

As $u(x,0) = 0$, we get

$C_2 e^{-px} C_3 = 0 \Rightarrow C_3 = 0$ [ because $C_2 \neq 0 \neq e^{-px}, \forall x$ ]

$U(x, y) = A e^{-px} \sin py$, $A = C_2C_4$

Now $u(x, a) = 0 \Rightarrow A e^{-px} \sin pa = 0 \Rightarrow \sin pa = 0$ [$\because A \neq 0$ ]

$\Rightarrow pa = n\pi, n \in I \Rightarrow p = n\pi / a, n = 0, \mp 1, ...$

$u(x, y) = \sum A_n e^{-n\pi x/a} \sin \frac{n\pi y}{a}$

$A_n$ being new constant.

This is the required solution in this case.

**Ex.** Show that the two dimensional Laplace equation $\nabla_1^2 V = 0$, in the plane polar coordinates r and $\theta$ has the solution of the form $(Ar^n + Br^n) e(\mp in\theta)$,

where A and B are n constants. Determine V if it satisfies

$\nabla_1^2 V = 0$ in the region $0 \leq r \leq a, 0 \leq \theta \leq 2\pi$ and

(i) V remains finite as $r \to 0$

(ii) $V = \sum_n Cn \cos(n\theta)$, on r = a.

**Sol:** Try yourself.

## Laplace equation in cylindrical coordinates:

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Let $V = R(r)\Theta(\theta)Z(z)$ be the solution

$$\frac{\partial V}{\partial r} = \Theta Z \frac{\partial R}{\partial r} \text{ and } \frac{\partial^2 V}{\partial r^2} = \Theta Z \frac{\partial^2 R}{\partial r^2}$$

$$\frac{\partial^2 V}{\partial \theta^2} = RZ \frac{\partial^2 \Theta}{\partial \theta^2}, \qquad \frac{\partial^2 V}{\partial z^2} = R\Theta \frac{\partial^2 Z}{\partial z^2}$$

$$\therefore \nabla^2 V = \Theta Z \frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \Theta Z \frac{\partial R}{\partial r} + \frac{1}{r^2} RZ \frac{\partial^2 \Theta}{\partial \theta^2} + R\Theta \frac{\partial^2 Z}{\partial z^2}$$

Or $$\frac{1}{R}\left[\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right] + \frac{1}{r^2 \Theta}\frac{d^2 \Theta}{d\theta^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0$$

Let $\frac{1}{Z}\frac{d^2 Z}{dz^2} = m^2$, *then* $\frac{d^2 Z}{dz^2} - m^2 Z = 0$

$$\therefore Z = e^{\pm mz}$$

Now let $\frac{1}{\Theta}\frac{d^2 \theta}{d\theta^2} = -n^2 \quad \Rightarrow \frac{d^2 \theta}{d\theta^2} + n^2 \Theta = 0 \qquad \therefore \Theta = e^{\pm in\theta}$

Now $$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + R\left(m^2 - \frac{n^2}{r^2}\right) = 0$$

which is Bessel's equation

its solution can be written as

R(r) = $A_n J_n(m_r) + B_n Y_n(m_r)$

Therefore V(r,$\theta$,z) = $\{A_n J_n(m_r) + B_n Y_n(m_r)\} e^{\pm im\theta} e^{\pm imz}$

## Home Assignments

**Exercise:** Solve the PDE

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

Subject to the condition $v = \frac{\partial \phi}{\partial r} = 0$ at r=a

And $v_r = U\cos\theta$, $\qquad v_\theta = {}^{1}\!/_{r}\dfrac{\partial\phi}{\partial\theta} = -U\sin\theta$ as r→ ∞

**Exercise:** Solve the BVP

$$\nabla^2 \text{u} = 0 0 \leq r \leq 10$$

$$0 \leq \theta \leq \pi$$

Subject to the conditions

$$u(10,\theta) = \frac{400}{\pi}(\pi\theta - \theta^2)$$

$$u(r,0) = 0 = u(r,\pi)$$

And $u(0,\theta)$ is finite

**Exercise:** Show that the solution of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = 0$$

Subject to the condition $u(x,0) = 0$

$$u_y(x,0) = \frac{1}{n}\sin nx$$

Where $n$ is a positive integer and

$$u(x,y) = \frac{1}{n^2}\sinh ny \sin nx$$

**Interior Dirichlet problem for a circle**

The Dirichlet problem for a circle is defined as follows:

To find the value of u at any point in the interior of the circle r = a in terms of its values on the boundary such that u is the single valued and continuous function within and on the circular region and satisfies the equation $\nabla^2 u = 0$ ; $0 \leq r \leq a$ subject to $u(a,\theta) = f(\theta)$ ; $0 \leq \theta \leq 2\pi$

We have

$\nabla^2 u = \partial^2 u / \partial r^2 + 1/r \frac{\partial u}{\partial r} + 1/r^2 \partial^2 u / \partial\theta^2 = 0$

We know that

$u(r,\theta) = \sum_{n=0}^{\infty}(C_n r^n + D_n r^{-n})\,(A_n' \cos n\theta + B_n' \sin n\theta\,)$

Since the function u is defined for all the values within and on the boundary of a circle

Therefore for r = 0 u(r,$\theta$ ) exists only for $D_n = 0\ \forall\ n$

Thus u(r,$\theta$ ) = $\sum_{n=0}^{\infty} r^n$ ($A_n\cos n\theta + B_n \sin n\theta$ )

= $a_0 / 2 + \sum_{n=1}^{\infty} r^n$ ($A_n\cos n\theta + B_n \sin n\theta$ ) where $A_0 = a_0 / 2$

= $a_0 / 2 + \sum_{n=1}^{\infty} r^n$ ($a_n\cos n\theta + b_n \sin n\theta$ )

Put u(a,$\theta$ ) = f($\theta$)

Therefore f($\theta$) = = $a_0 / 2 + \sum_{n=1}^{\infty} a^n$ ($a_n\cos n\theta + b_n \sin n\theta$ )

This is the full range Fourier series ( i.e. $a_n, b_n \neq 0$ )
Now $a_0 = 1/\pi \int_0^{2\pi} f(\varphi)d\varphi$

$a_n = 1/\pi a^n \int_0^{2\pi} f(\varphi) \cos n\varphi\, d\varphi$

$b_n = 1/\pi a^n \int_0^{2\pi} f(\varphi) \sin n\varphi\, d\varphi$

u(r,$\theta$ )=$1/2\pi \int_0^{2\pi} f(\varphi)d\varphi + 1/\pi \sum_{n=1}^{\infty}(r/a)^n \int_0^{2\pi} \cos n\theta \cos n\varphi + \sin n\theta \sin n\varphi d\varphi$

u(r,$\theta$ )=$1/\pi$ { $\int_0^{2\pi} f(\varphi)$ [$1/2 + \sum_{n=1}^{\infty}(r/a)^n \cos n\theta \cos n\varphi + \sin n\theta \sin n\varphi$]$d\varphi$ }

u(r,$\theta$ )=$1/\pi$ { $\int_0^{2\pi} f(\varphi)$ [$1/2 + \sum_{n=1}^{\infty}(r/a)^n \cos n(\varphi - \theta)$]$d\varphi$ } ...(1)
let c = $\sum_{n=1}^{\infty}(r/a)^n \cos n(\varphi - \theta)$
S = $\sum_{n=1}^{\infty}(r/a)^n \sin n(\varphi - \theta)$

C + is = $\sum_{n=1}^{\infty} \{(r/a)\, e^{i(\varphi-\theta)}\}^n$

S = r/a $e^{i(\varphi-\theta)}$ / 1- r/a $e^{i(\varphi-\theta)}$ ; r/a < 1 and | $e^{i(\varphi-\theta)}| \leq 1$

S = r/a [$\cos(\varphi - \theta) + i \sin(\varphi - \theta)$] / 1- r/a ( $\cos(\varphi - \theta) + i \sin(\varphi - \theta)$]

Equating real and imaginary parts

C = r/a $[\cos(\varphi-\theta) \pm {r^2}/{a^2}]$ / 1-(2 r/a) ( $\cos(\varphi-\theta) + {r^2}/{a^2}]$

Using in (1) we get

$$u(r,\theta)=\frac{1}{2\pi}\{\int_0^{2\pi} f(\varphi)\,[1/2+\frac{\frac{r}{a}\cos(\varphi-\theta)+\frac{r^2}{a^2}}{1-2\left(\frac{r}{a}\right)\cos(\varphi-\theta)+\frac{r^2}{a^2}}]\}$$

$$u(r,\theta)=\frac{1}{2\pi}\{\int_0^{2\pi}\frac{(a^2-r^2)}{a^2-2ar\cos(\varphi-\theta)}f(\varphi)\,d\varphi$$

This is the Poisson's integral formula for a circle.

**Exterior Dirichlet Problem for a circle:**

The exterior Dirchelet's problem is described by

$$\nabla^2\phi = 0\, 0 \le \theta \le 2\pi$$

with $\phi(a,\theta)=f(\theta)$ $\qquad 0\le\theta\le 2\pi$ at r=a

where $f(\theta)$ is a continuous function of $\theta$ on the surface $r=a$ and $\varphi$ must be bounded as $r \to \infty$.

The solution is of the form

$$\phi(r,\theta)=\sum_{n=0}^{\infty}\left(C_n r^n + D_n r^{-n}\right)\left(A_{n'}\cos n\theta + B_{n'}\sin n\theta\right)$$

As r→ ∞ ; $\emptyset(r,\theta)$ exists finitely

$$\therefore C_n = 0 \forall n$$

$$\phi(r,\theta)=\sum_{n=0}^{\infty} r^{-n}\left(A_n\cos n\theta + B_n\sin n\theta\right)$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty} r^{-n}\left(a_n\cos n\theta + b_n\sin n\theta\right) \qquad ...(1)$$

Now by the given condition

$$f(\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a^{-n}\left(a_n\cos n\theta + b_n\sin n\theta\right) \qquad ...(2)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\psi)d\psi$$

$$a_n = \frac{a^n}{\pi}\int_0^{2\pi} f(\psi)\cos(n(\psi))d\psi$$

$$b_n = \frac{a^n}{\pi}\int_0^{2\pi} f(\psi)\sin(n(\psi))d\psi$$

Therefore we have from (1)

$$\phi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\psi)d\psi + \sum_{n=1}^{\infty}\int_0^{2\pi}\left(\frac{a^n}{r}\right)\frac{1}{\pi}\left[\cos n\theta\cos n\psi + \sin n\theta\sin n\psi\right]f(\psi)d\psi$$

$$\phi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\psi)d\psi + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\int_0^{2\pi}\frac{1}{\pi}\left[\cos n(\psi-\theta)\right]f(\psi)d\psi$$

$$\phi(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} f(\psi)\{\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\left[\cos(\psi-\theta)\right]\}d\psi \qquad \ldots(3)$$

Let C = $\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \cos n(\psi-\theta)$

And S = $\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \sin n(\psi-\theta)$

Therefore $C + iS = \sum_{n=1}^{\infty}\left[\left(\frac{a}{r}\right)e^{i(\psi-\theta})\right]^n$

$$= \frac{\frac{a}{r}e^{i(\psi-\theta)}}{1-\frac{a}{r}e^{i(\psi-\theta)}}$$

Now by rationalising and comparing real parts on both sides we get

$$C = \frac{\frac{a}{r}\cos(\psi-\theta) - \frac{a^2}{r^2}}{1 - \frac{2a}{r}\cos(\psi-\theta) + \frac{a^2}{r^2}}$$

$$\phi(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} f(\psi)\{\frac{1}{2} + \frac{\frac{a}{r}\cos(\psi-\theta) - \frac{a^2}{r^2}}{1 - \frac{2a}{r}\cos(\psi-\theta) + \frac{a^2}{r^2}}\}d\psi \qquad \text{...(4)}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2 - a^2)f(\psi)d\psi}{r^2 - 2ar\cos(\psi-\theta) + a^2}$$

**INTERIOR NEUMANN PROBLEM FOR A CIRCLE**

The interior Neumann problem for a circle is defined as follows:

To find the value of U at any point in the interior of the circle r=a such that

$\nabla^2 u = 0;\ 0 \le r < a; \qquad 0 \le \theta \le 2\pi$

And $\frac{\partial u}{\partial n} = \frac{\partial u(r,\theta)}{\partial r} = g(\theta)$ on r=a

By the method of separation of variable the general solution of the given equation is given by

$$u(r,\theta) = \sum_{n=0}^{\infty}(C_n r^n + D_n r^{-n})(A_n \cos n\theta + B_n \sin n\theta)$$

At r=0 the solution u should be finite and therefore D$_n$=o ∀ $n$

Therefore $u(r,\theta) = \sum_{n=0}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta)$

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta)$$

Therefore $\frac{\partial u}{\partial r} = \sum_{n=1}^{\infty} n r^{n-1}(a_n \cos n\theta + b_n \sin n\theta)$

Now $\dfrac{\partial u(r,\theta)}{\partial r} = g(\theta)$ where

$$a_n = \frac{1}{na^{n-1}\pi}\int_0^{2\pi} g(\phi)\cos n\phi d\phi$$

$$b_n = \frac{1}{na^{n-1}\pi}\int_0^{2\pi} g(\phi)\sin n\phi d\phi$$

Therefore

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{r^n}{n\pi a^{n-1}} \int_0^{2\pi} g(\phi)[\cos n\phi \cos n\theta + \sin n\phi \sin n\theta] d\phi$$

**Home Assignment**

**Exercise:** By separating the variables, show that the equation $\nabla^2 \mathrm{v} = 0$ has a solution of the form $A\exp(\pm nx \pm iny)$ where A and n are constants

Deduce that the function is of the form

$$v(x,y) = \sum_r A_r e^{\frac{-r\pi x}{a}} \sin\left(\frac{r\pi y}{a}\right) \qquad x \geq 0; \quad 0 \leq y \leq a$$

Where $A_r's$ being constants are plane harmonic functions satisfying the conditions

$$v(x,0) = 0, \quad v(x,a) = 0, \quad v(x,y) \to 0, \quad as\ x \to \infty$$

**Exercise:** A thin rectangular homogeneous thermally conducting plane occupies the region $0 \leq y \leq b,\ 0 \leq x \leq a$. The edge $y = 0$ is held at temperature $t(x)(x-a)$, where $T$ is a constants and other edges are maintained at $'0'$. The other faces are insulated and there is no heat source or sink inside the plate. Find the steady state temperature inside the plate.

**Sol.** Pleases try yourself.

## PARABOLIC DIFFERENTIAL EQUATIONS

The have equation of the form $Rr + Ss + Tt + f(x,y,z,p,q) = 0$ with $S^2 - 4RT = 0$ is known as parabolic differential equation. The diffusion phenomenon such as conduction heat in solids and diffusion of viscous fluid flow as generated by a PDE of parabolic type.

The general equation for heat transfer is governed by the following equations

$$\frac{\partial T}{\partial t} = k\nabla^2 \mathrm{T}$$

Where $\frac{\partial T}{\partial t}$ is the time derivative and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ represents the derivative w.r.t., space.

**Heat Equation:** The heat conduction equation $\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$

May have numerous solutions unless a set of initial and boundary conditions are satisfied. The boundary conditions are mainly of three types and are briefly given below.

**Boundary condition I:** The temperature is prescribed all over the boundary surface. This type of boundary condition depends on the problem under investigation. Some times the temperature on the boundary surface is a function of position only or is a function of time only or a constant. A special case includes T(r, t ) = 0 on the surface of boundary, which is called a homogenous boundary condition.

**Boundary condition II:** The flux of heat, i.e. the normal derivative of temperature $\frac{\partial T}{\partial n}$ is prescribed on the surface of boundary. This is called the Neumann condition. A special case includes $\frac{\partial T}{\partial n}$ =0 on the boundary.This homogenous boundary condition is also called insulated boundary condition which states that the heat flow across the surface is zero.

**Boundary condition III:** A linear combination of the temperature and the heat flux is prescribed on the boundary

i.e. $\mathrm{K}\frac{\partial T}{\partial n} + \mathrm{ht} = \mathrm{G}(x, \mathrm{t})$

this type of boundary condition is called Robins condition. It means that the boundary surface dissipates heat by convection. By Newton's law of cooling, we have

$\mathrm{K}\frac{\partial T}{\partial n} = \mathrm{h}(\mathrm{T} - \mathrm{T}_{\mathrm{a}})$

$\mathrm{T}_{\mathrm{a}}$ is the temperature of surrounding

Its special case may be taken as

$$K\frac{\partial T}{\partial n} + hT = 0$$

Which is homogenous boundary condition.

The other boundary conditions such as the heat transfer due to radiation obeying the fourth power temperature law and these associated with change of phase like melting, ablation etc. give rise to non linear boundary conditions.

## SEPARATION OF VARIABLE METHOD:

We consider the one dimensional heat conduction equation

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \qquad ...(1)$$

let $T(x,t) = X(x)\,Y(t)$ ...(2)

be the solution of the differential equation (1) substituting from (2) into (1) we obtain

$\frac{X''}{X} = \frac{1}{K}\frac{Y'}{Y} = \lambda$ (separation parameter) then we have

$$\frac{d^2X''}{dx^2} - \lambda X = 0 \qquad ...(3)$$

$$\frac{dY}{dt} - K\lambda Y = 0 \qquad ...(4)$$

In Solving equations (3) and (4) three distinct cases arise.

**Case I:** Let-$\lambda$>0, say $\alpha^2$ the solution will have the form

$X = C_1e^{\alpha x} + C_2e^{-\alpha x}$, $Y = C_3e^{\alpha^2 kt}$ ...(5)

**Case II:** let $\lambda = -\alpha^2$, $\alpha$ is positive, then solution will have the form

Which provide us $X = C_1\cos\alpha x + C_2\sin\alpha x$, and $Y = C_3e^{-\alpha^2 kt}$ ...(6)

**CASE III:** let $\lambda = 0$ then we have

$$X = C_1x + C_2,\ Y = C_3 \qquad ...(7)$$

Thus various possible solutions of the one dimensional heat conduction equation (1) are

$T(x,t) = (Ae^{\alpha x} + Be^{-\alpha x})\,e^{k\alpha^2 t}$

$T(x,t) = (A\cos\alpha x + B\sin\alpha x)\,e^{-\alpha^2 kt}$ ...(8)

$T(x,t) = (Ax + B)$ where $A = C_1C_3$, $B = -C_2C_3$

**Example 2:** Show that the solution of the equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \qquad \ldots(1)$$

Satisfying the conditions

(1) $T \to 0$, as $t \to \infty$
(2) $T = 0$, for $x = 0$ and $x = a$ for all $t > 0$
(3) $T = x$, when $t = 0$ and $0 < x < a$ is

$T(x,t) = 2a/\pi \sum_{n=1}^{\infty}(-1)^{n-1}/n)\sin(\frac{n\pi}{a}x)\exp[-(n\pi/a)^2 t]$

**Solution:** we know that the solution of (1) is

[1] $T(x,t) = (Ae^{\alpha x} + Be^{-\alpha x})\exp(\alpha^2 t)$
[2] $T(x,t) = (A\cos\alpha x + B\sin\alpha x)\, exp\,(\alpha^2 t)$
[3] $T(x,t) = Ax + B$

Clearly solutions represented by (1) and (2) does not satisfy the given conditions. Therefore the most feasible solution for the equation (1) can be treated (2)

$T(x,t) = (A\ \cos\alpha x + B\ \sin\alpha x)\, e^{-\alpha^2 t}$using the boundary condition (2) we have

$0 = [A(1) + B(0)]\, e^{-\alpha^2 t}$

Or $0 = Ae^{-\alpha^2 t}$

Or $A = 0$

Also $T(0,t) = 0 = (B \sin \alpha a)e^{-\alpha^2 t}$

Since $B \neq 0$ and $e^{-\alpha^2 t} \neq 0$

$\Rightarrow \quad \sin \alpha a = 0$

$\Rightarrow \alpha a = n\pi \quad$ or $\quad \alpha = n\pi/a$

Hence the solution is of the form

$T(x,t) = B\sin\left(\frac{n\pi}{a}x\right)e^{-\alpha^2 t}$

$\quad = B\sin\left(\frac{n\pi}{a}x\right)\exp\left(-\frac{n^2\pi^2}{a^2}t\right)$

Since the heat conduction equation is linear therefore the most general solution is obtained by applying the principle of superposition

i.e. $T(x, t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{a}x\right) \exp\left(-\frac{n^2\pi^2}{a^2}t\right)$

now using condition (3) we get

$T = x$ for $t = 0$

$\Rightarrow x == \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{a}x\right) \times 1$ $(\because t = 0)$

Which is a half range Fourier sine series therefore $B_n = 2/a \int_0^a x\sin(\frac{n\pi}{a}x)\, dx$

Let $\frac{n\pi}{a}x = z$

$\frac{n\pi}{a}dx = dz$

For $x = 0$, $z = 0$

For $x = a$, $z = n\pi$

Therefore $B_n = \frac{2}{a}\int_0^{n\pi} \frac{a^2}{n\pi^2} z\sin z dz$

$= \frac{2a}{\pi}\frac{-1^{n+1}}{n}$

Therefore $T(x, t) = \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{a}x\right) \exp\left(-\frac{n^2\pi^2}{a^2}t\right)$

**EX:** The ends A and B of a rod, $10\ cm$ in length are kept at temperature $0^0$c and $100^0$ c respectively until the steady state conditions prevails. Suddenly the temperature at the end A is increased to $20^0$ c and the end B is decreased to $60^0$ c. Find the temperature distribution in rod at time at t.

**Sol**. The problem is described by

$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$ ; $0<x<10$

Subject to the conditions

$$T(0, t) = 10$$

$$T(10, t) = 100$$

For steady state $\frac{d^2T}{dx^2} = 0$

Which implies that $T_s = Ax + B$

Now for x = 0, T= 0 implies that B = 0, Therefore $T_s = Ax$

And for x = 0, T = $100^0$ C, implies that A = 10

Thus the initial steady temperature distribution in rod is

$T_s(x) = 10x$

Similarly when the temperature at the ends A and B are changed to $20^0$c and $60^0$C, the final steady temperature in rod is

$T_s(x) = 4x + 20$

Which will be attained after long time. At any instant of time the temperature

$T\ (x, t)$ in rod is given by

$$T(x,t) = T_t(x,t) + T_s(x)$$

Where$T_t(x,t)$ is the transient temperature distribution which tends to zero as $t \to \infty$. Now $T(x,t)$ satisfies the given partial differential equation. Hence its general solution is of the form

$$T(x,t) = T_t(x,t) + T_s(x)$$

$$T(x,t) = 4x + 20 + e^{-K\lambda^2 t}\,(B\cos\lambda x + C\sin\lambda x)$$

For x = 0, T = $20^0$C, we obtain

$20 = 20 + B\, e^{-K\lambda^2 t} \Rightarrow B = 0, \quad t > 0$

For x = 10, T = $60^0$ we get

$60 = 60 + e^{-K\lambda^2 t} C \ \sin 10\,\lambda$

$\Rightarrow \sin 10\,\lambda = 0 \quad \Rightarrow \quad \lambda = \frac{n\pi}{10}, \quad n \epsilon\ I$

The principle of superposition yields

$$T(x,t) = 4x + 20 + \sum_{n=1}^{\infty} C_n \exp\left[-k\left(\frac{n\pi}{10}\right)^2 t\right]\sin\left(\frac{n\pi}{10}x\right)$$

using the initial condition $T = 10\,x$, when t= 0, we obtain

$$10x = 4x + 20 + \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{10}x\right)$$

Where $C_n = \frac{2}{10}\int_0^{10}(6x-20)\sin\left(\frac{n\pi}{10}x\right)dx$

$$= \frac{-1}{5}\left[(-1)^n\frac{800}{n\pi} - \frac{200}{n\pi}\right]$$

Thus the required solution is

$$T(x,t) = 4x + 20 - \frac{1}{5}\sum_{n=1}^{\infty} \left[(-1)^n\frac{800}{n\pi} - \frac{200}{n\pi}\right] \exp\left[-k\left(\frac{n\pi}{10}\right)^2 t\right]\sin\left(\frac{n\pi}{10}x\right)$$

**Diffusion equation in cylindrical coordinates**

Consider a three dimensional diffusion equation

$$\frac{\partial T}{\partial t} = K\nabla^2 T$$

In cylindrical coordinates $(r,\theta,z)$ it become

$$\frac{1}{k}\frac{\partial T}{\partial t} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \qquad ...(1)$$

We assume separation of variables in the form

$$T(r,\theta,z) = R(r)\,\Theta(\theta)Z(z)\emptyset(t)$$

Substituting this in (1), we get

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{z''}{z} = \frac{1}{k}\frac{\phi'}{\phi} = -\lambda^2$$

Where $-\lambda^2$ is a separation parameter.

Then $\phi' + k\lambda\phi^2 = 0$ ...(2)

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \lambda^2 = -\frac{z''}{z} = -\mu^2 \text{ (say)}$$

The equation in Z, R and $\Theta$ becomes

$$z'' - \mu^2 z = 0 \qquad ...(3)$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \left(\lambda^2 + \mu^2\right)r^2 = -\frac{\Theta''}{\Theta} = \Omega^2$$

Therefore

$$\Theta'' + \Omega^2\ \Theta = 0 \qquad ...(4)$$

And $$R'' + \frac{1}{r}R' + \left(\lambda^2 + \mu^2 - \frac{\Omega^2}{r^2}\right)R = 0 \qquad ...(5)$$

Equations (2) and (4) have particular solutions of the form

$\phi = e^{-k\lambda^2 t}$

$\Theta = (C\cos\Omega\theta + D\sin\Omega\theta)$

$z = Ae^{\mu z} + Be^{-\mu z}$

Equation (5) is Bessel's equation of order $\Omega$ and its general solution is

$$\text{R(r)} = C_1 J_\Omega\left(\sqrt{(\lambda^2 - \mu^2)}r\right) + C_2 Y_\Omega\left(\sqrt{(\lambda^2 + \mu^2)}r\right)$$

Where $J_\Omega(r) and Y_\Omega(r)$ are Bessel functions of order $\Omega$ of first and second kind respectively. Equation (5) is singular for r = 0, the physically meaningful solution must be twice continuously differentiable in $0 \le r \le a$.

Hence equation (5) has only one bounded solution

i.e. $\text{R(r)} = J_\Omega\left(\sqrt{\left(\lambda^2 - \mu^2\right)}r\right)$

Finally the general solution of equation (1) is given as

$$\text{T}(r,\theta,z,t) = \exp\left(-K\lambda^2 t\right)\left[Ae^{\mu z} + Be^{-\mu z}\right]\left[(C\cos\Omega\theta + D\sin\Omega\theta)\,j_\Omega\sqrt{(\lambda^2 - \mu^2)r}\right]$$

...(6)

**Assignment**

**EX:** Find the solution of the diffusion equation

$$\frac{\partial T}{\partial t} = \text{K}\nabla^2 T$$

**Ex:** A uniform rod of length 1 with thermally insulated surface is initially at temperature $\theta = \theta_0$ At t=0, one end is suddenly cooled to $\theta = 0^0 C$

And subsequently maintained at this temperature, the other end remains thermally insulated. Find the temperature distribution $\theta(x,t)$.

**EX:** Find the solution of the 1-D diffusion equation satisfying the following conditions

(i) $T$ is bounded as $t \to \infty$

(ii) $\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0, \ \ \forall\, t$

(iii) $T(x, 0) = x(a - x); \qquad 0 < x < 0$

**EX:** Solve the boundary value problem

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < t$$

Subject to the conditions

(i) $\frac{\partial u(0,t)}{\partial x} = 0$

(ii) $\frac{\partial u(l,0)}{\partial x} = 0$

(iii) $u(x, 0) = x$

**EX:** Solve the following equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Subject to the conditions

(i) $u(x, 0) = 3 \sin n\pi x$

(ii) $u(0, t) = 0 = u(l, t), \qquad 0 < x < l, \quad t > 0.$

**EX:** Find the solution of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Subject to the conditions

(i) $u(x, 0) = 3 \sin n\pi x$

(ii) $u(0, t) = 0 = u(l, t), \qquad 0 < x < l, \quad t > 0.$

**EX:** Find the solution of the equation

$$\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2}$$

Subject to the conditions

(i) $v = v_0 \sin nt$ where $x = 0 \quad \forall t$

(ii) $v = 0 \qquad x \to \infty$

## HYPERBOLIC DIFFERENTIAL EQUATIONS:

One of the most important and typical homogenous hyperbolic differential equation is the wave equation of the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Where $C$ is the wave speed.

The differential equation above is used in many branches of physics and engineering and is seen in many situations such as transverse vibrations in strings or membrane, longitudinal vibrations in a bar, propagation of sound waves, electromagnetic waves, sea waves, elastic waves in solid and surface waves in earth quakes.

The solution of wave equations are called wave functions.

Remark: The Maxwell's equations of electromagnetic theory is given by

$$\nabla . E = 4\pi P$$

$$\nabla . \vec{H} = 0$$

$$\nabla \times \vec{E} = \frac{-1}{C}\frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \frac{4\pi i}{C} + \frac{1}{C}\frac{\partial \vec{t}}{\partial t}$$

Where E is an electric field, $\rho$ is electric charge density, H is the magnetic field, C is the current density and C is the velocity of light.

**Exercise :** show that in the absence of a charge, the electric field and the magnetic field in the Maxwell's equation satisfy the wave equation.

**Solution:** we have

$$\text{Curl } \vec{E} = \quad \nabla \times \vec{E} = \frac{-1}{C}\frac{\partial \vec{H}}{\partial t}$$

$$\text{Consider } \nabla \times (\nabla \times E) = \nabla \times \left( \frac{-1}{C}\frac{\partial \vec{H}}{\partial t} \right)$$

$$= \frac{-1}{C}\frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right)$$

$$\text{This implies } \nabla \times (\nabla \times E) = \frac{-1}{C^2}\frac{\partial^2 E}{\partial t^2}$$

But $\nabla \times (\nabla \times E)$ Can be expressed as

$$\nabla(\nabla \cdot E) - \nabla^2 E = \nabla(4\pi\rho) - \nabla^2 E$$

$$= -\nabla^2 E$$

$$-\nabla^2 E = \frac{-1}{C^2}\frac{\partial^2 E}{\partial t^2}$$

Or $\frac{\partial^2 E}{\partial t^2} = C^2 \ \nabla^2 E$

Which is a wave equation satisfied by $\vec{E}$

Similarly we can observe the magnetic field H satisfies the wave equation $\frac{\partial^2 H}{\partial t^2} = c^2 \nabla^2 H$ .

**Solution of Wave equation: (Method of separation of variables)**

We have $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

Let U(x, t) = X(x) T(x) be the solution of (1)

$\therefore \frac{\partial^2 U}{\partial t^2} = XT''$ and $\frac{\partial^2 U}{\partial x^2} = TX''$

Using in (1),we get

$XT'' = C^2 TX'' \Rightarrow \frac{T''}{T} = C^2 \frac{X''}{X} = \lambda$ (say)

Where $\lambda$ is a separation parameter

$$\Rightarrow T'' - \lambda T = 0 \qquad ...(2)$$

And $$C^2 X'' - \lambda X = 0 \qquad ...(3)$$

T = $Ae^{\sqrt{\lambda t}} + Be^{-\sqrt{\lambda t}}$

CASE I: If $\lambda > 0$ say $\lambda = k^2$

Therefore $$T = Ae^{kt} + Be^{-kt} \qquad ...(4)$$

Similarly

$C^2 X'' - K^2 X = 0$

$X(x) = D e^{\frac{K}{C} X} + E e^{-\frac{K}{C} X}$

$$\therefore U(x,t) = (Ae^{Kt} + Be^{-Kt})(De^{\frac{K}{C}x} + Ee^{-\frac{K}{C}x})$$

This is the required solution

Case II: if $\lambda = 0$

Then $T'' = 0$ and $C^2X''=0$

$\Rightarrow T = Ct + D$ and $X = Ax + B$

Therefore $U(x,t) = (Ax + B)(Ct + D)$

Case III: If $\lambda < 0$ say $\lambda = -K^2$

$$\Rightarrow \frac{T''}{T} = -K^2 \text{ and } \frac{C^2X''}{X} = -K^2$$

$\Rightarrow T'' + K^2T = 0$ and $C^2X'' + K^2X = 0$

So T = $(A\cos Kt + B\sin Kt)$ and X = $\left(D\cos\frac{K}{C}x + E\sin\frac{K}{C}x\right)$

Therefore U(x, t) = $(A\cos Kt + B\sin Kt)\left(D\cos\frac{K}{C}x + E\sin\frac{K}{C}x\right)$

**REMARK:** From the above solutions of the wave equation for $0 \leq x \leq l$ and $t > 0$

Subject to the conditions

$U(0,t) = 0\,;\ t > 0,$

$$U(l,t) = 0$$

Using the conditions in **case I**

0 = U (0,t) = $\left(Ae^{Kt} + Be^{Kt}\right)(D + E)$

$\Rightarrow$ D+E = 0 ...(5)

Now U(l, t) =0

$$\Rightarrow U(l,t) = \left(Ae^{Kt} + Be^{Kt}\right)\left(De^{\frac{K}{C}l} + Ee^{\frac{K}{C}l}\right) = 0$$

$$\Rightarrow \left( De^{\frac{K}{C}l} + Ee^{\frac{K}{C}l} \right) = 0$$

$$\Rightarrow De^{2\frac{K}{C}l} + E = 0 \qquad \ldots(6)$$

$\Rightarrow e^{2\frac{K}{C}l} = 1$ By comparing coefficients in (5) and (6)

$$\Rightarrow 2\frac{K}{C}l = 0 \Rightarrow \frac{K}{C}l = 0$$

Either $l = 0$ or $\frac{K}{C} = 0$

Therefore solution in case (1) is not acceptable

Now using in case II we get

0=U (0, t) =(C t+ D)B

$\Rightarrow B = 0$

And 0 $= U(l,t) = (Al + B)(Ct + D)$

$\Rightarrow (Al + B)(Ct + D) = 0$

Implies A =0

Implies A=0=B

Now using the conditions in case III

U(x, t) $= (A\cos Kt + B\sin Kt)\left( D\cos\frac{K}{C}x + E\sin\frac{K}{C}x \right)$

Now U (0,t) =0

$\Rightarrow (A\cos Kt + B\sin Kt)(D) = 0$

$\Rightarrow$ D =0

Also 0 $= U(l,t) = (A\cos Kt + B\sin Kt)\left( E\sin\frac{K}{C}l \right)$

$$\Rightarrow\left(E\sin\frac{K}{C}l\right)=0$$

$$\Rightarrow\left(\frac{K}{C}l\right)=n\pi$$

$$\Rightarrow\frac{K}{C}=\frac{n\pi}{l}$$

$$\therefore U(x,t)=\left(A\cos Kt+B\sin Kt\right)\left(E\sin\frac{n\pi}{l}x\right)$$

Therefore by using superposition principle

$$U_n(x,t)=\sum E_n\sin\left(\frac{n\pi}{l}x\right)\left(A_n\cos\left(\frac{cn\pi}{l}\right)t+B_n\sin\left(\frac{cn\pi}{l}\right)t\right)$$

**Ex:** By the separation of variables, show that one dimensional wave equation

$$\frac{\partial^2 Z}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 Z}{\partial t^2}$$

Has solution of the form $A\exp\left(\pm inc\pm inct\right)$
Where A and n are constants. Hence show that the function of the form

$$Z(x,t)=\sum_r\left(A_r\cos\left(\frac{r\pi ct}{a}\right)+B_r\sin\left(\frac{r\pi ct}{a}\right)\right)\sin\left(\frac{r\pi x}{a}\right)$$

Where $A_{r\cdot s}$ and $B_{r\cdot s}$ are constants, satisfying the wave equation and the boundary conditions

$$Z(0,t)=0=Z(a,t)\,;\;t>0$$

Ex: Obtain the solution of the radio equation

$$\frac{\partial^2 V}{\partial x^2}=LC\frac{\partial^2 V}{\partial t^2}$$

Appropriate to the case when the periodic e.m.f. $V_0\cos\left(pt\right)$ is applied at the end x=0 of the line.

**Exercise:** A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given

$$y=y_0\sin^3\left(\frac{\pi x}{l}\right)$$

It is released from rest from this position.
Find the displacement $y(x,t)$

**Sol.** We have the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{C^2}\frac{\partial^2 y}{\partial t^2}$$

Such that $y(0,t) = 0 = y(l,t)$

And $y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$

And at t=0

$$\frac{dy}{dt} = 0$$

Let y(x, t) = X(x) T(t) be the solution
Then

$$\frac{\partial^2 y}{\partial x^2} = T\frac{\partial^2 X}{\partial t^2} = TX''$$

*and* $$\frac{\partial^2 y}{\partial x^2} = X\frac{\partial^2 T}{\partial t^2} = XT''$$

$\therefore$ We have

$$TX'' = \frac{1}{C^2}XT''$$

Therefore

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda ct + D\sin\lambda ct)$$

Now,

$$y(0,t) = 0$$

$$\Rightarrow A(C\cos\lambda ct + D\sin\lambda ct) = 0$$

$$\Rightarrow A = 0$$

and $y(x,t) = (B\sin\lambda x)(C\cos\lambda ct + D\sin\lambda ct)$

And $$\frac{\partial y(x,0)}{\partial t} = 0$$

$$0=\left(\frac{\partial y}{\partial t}\right)_{t=0}=[(A\cos\lambda x+B\sin\lambda x)(\lambda cD\cos\lambda ct-\lambda cC\sin\lambda ct)]_{t=0}$$

$$\Rightarrow (A\cos\lambda x+B\sin\lambda x)(\lambda cD)=0$$

$$\Rightarrow D=0$$

Thus

$$y(x,t)=E\sin\lambda x\cos\lambda ct$$

Where $E = Bc$

$$\Rightarrow \quad E\sin\lambda l\cos\lambda ct=0$$

$$\Rightarrow \quad \sin\lambda l=0$$

$$\Rightarrow \quad \lambda l=n\pi$$

$$\Rightarrow \quad \lambda=\frac{n\pi}{l}$$

$$\therefore \quad y(x,t)=\sum_n E_n\sin\left(\frac{n\pi}{l}x\right)\cos\left(\frac{n\pi}{l}ct\right)$$

By the given condition

$$y(x,0)=y_0\sin^3\left(\frac{\pi}{l}x\right)$$

Therefore

$$y_0\sin^3\left(\frac{\pi}{l}x\right)=\sum_n E_n\sin\left(\frac{n\pi}{l}x\right)$$

$$=E_1\sin\left(\frac{\pi}{l}x\right)+E_2\sin\left(\frac{2\pi}{l}x\right)+E_3\sin\left(\frac{3\pi}{l}\right)x+\dots$$

We know that

$$\sin^3 x=\frac{3\sin x-\sin 3x}{4}$$

Or

$$y_0\left[\frac{3\sin\left(\frac{\pi x}{l}\right)-\sin\left(\frac{3\pi x}{l}\right)}{4}\right]=E_1\sin\left(\frac{\pi x}{l}\right)+E_2\sin\left(\frac{2\pi x}{l}\right)+E_3\sin\left(\frac{3\pi x}{l}\right)+\ldots$$

Now comparing the coefficients on both sides we get

$$E_1=\frac{3y_0}{4},\ E_2=0,\ E_3=\frac{-y_0}{4},\quad E_4=E_5=\ldots=0$$

$$\therefore\ y(x,t)=\frac{3y_0}{4}\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi}{l}\right)ct-\frac{y_0}{4}\sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{\pi}{l}\right)ct$$

This is the required solution.

## PERIODIC SOLUTION IN CYLINDERICAL COORDINATES:

In cylindrical coordinates with u depending only on r. The one dimensional wave equation assume the form.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=\frac{1}{C^2}\frac{\partial^2 u}{\partial t^2} \qquad \ldots(1)$$

Assume that

$$U=F(r)e^{iwt}$$

Acts as a solution

$$\frac{\partial U}{\partial r}=F'(r)e^{iwt}$$

$$-\ r\frac{\partial U}{\partial r}=rF'(r)e^{iwt}$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=F'(r)e^{iwt}+rF''(r)e^{iwt}$$

$$\frac{\partial^2 u}{\partial t^2}=-w^2F(r)e^{iwt}$$

Now substituting in (1)we get

$$\frac{1}{r}\left(F'(r)e^{iwt}+rF''(r)e^{iwt}\right)=\frac{1}{C^2}\left(-w^2F(r)e^{iwt}\right)$$

$$\Rightarrow \quad F''(r)+\frac{F'(r)}{r}+\frac{w^2}{c^2}F(c)=0$$

Which is a form of Bessel's equation and hence we have

$$F=AJ_o\left(\frac{wr}{c}\right)+By_o\left(\frac{wr}{c}\right)$$

In complete form we can write this equation as

$$F=C_1\left[J_o\left(\frac{wr}{c}\right)+iy_o\left(\frac{wr}{c}\right)\right]+C_2\left[J_o\left(\frac{wr}{c}\right)-iy_o\left(\frac{wr}{c}\right)\right]$$

Therefore the complete solution for the periodic function is

$$U=\left\{AJ_o\left(\frac{wr}{c}\right)+By_o\left(\frac{wr}{c}\right)\right\}e^{iwt}$$

## Cauchy problem for inhomogeneous wave equation:

The wave equation

$$\frac{\partial^2 u}{\partial t^2}=C^2\frac{\partial^2 u}{\partial x^2}+f(x,t)$$

Subject to the initial conditions

$$u(x,0)=\eta(x), \quad \frac{\partial u(x,0)}{\partial t}=v(x) \qquad ...(1)$$

and $$\frac{\partial^2 u}{\partial t^2}-C^2\frac{\partial^2 u}{\partial x^2}=f(x,t) \qquad ...(2)$$

Subject to the homogenous initial conditions

$$u_2(x,0)=0 \quad \text{and} \quad \frac{\partial u_2(x,0)}{\partial t}=0 \qquad ...(3)$$

Integrating equation (2) over the region we get

$$\iint_R \left(\frac{\partial^2 u}{\partial t^2}-C^2\frac{\partial^2 u}{\partial x^2}\right)dxdt=\iint_R f(x,t)dxdt$$

Using Greens theorem in plane we get

$$-\iint_R \frac{\partial}{\partial x}\left(C^2\frac{\partial u_2}{\partial x}\right)-\frac{\partial}{\partial t}\left(\frac{\partial u_2}{\partial t}\right)dxdt=\iint_R f(x,t)dxdt$$

$$\Rightarrow \quad -\oint_R \left(\frac{\partial u_2}{\partial t}dt+c^2\frac{\partial u_2}{\partial x}dx\right)=\iint_R f(x,t)dxdt \qquad . \; . \; . \quad (4)$$

Where 2R denotes the boundary of the region R. The boundary $\partial R$ comprises of three segments PB, PA and AB

Along PB, $\quad \frac{dx}{dt}=-c$

And along PA, $\quad \frac{dx}{dt}=c$

Using these, we have from equation (4)

$$\oint_{BP} C\left(\frac{\partial u_2}{\partial t}dt+\frac{\partial u_2}{\partial x}dx\right)-\oint_{PA} C\left(\frac{\partial u_2}{\partial t}dt+\frac{\partial u_2}{\partial x}dx\right)=\iint_R f(x,t)dxdt$$

We know that for any function Z=Z(x, y)

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$$

$$\Rightarrow \quad \oint_{BP} Cdu_2-\oint_{PA} Cdu_2=\iint_R f(x,t)dxdt$$

$$\Rightarrow \quad Cu_2(P)-Cu_2(B)-Cu_2(A)+Cu_2(P)=\iint_R f(x,t)dxdt$$

Using conditions given in (3), we have

$$u_2(A)=u_2(B)=0$$

Therefore we have

$$2Cu_2(P)=\iint_R f(x,t)dxdt$$

$$\Rightarrow \quad u_2(P)=\frac{1}{2C}\int_0^{t_0}\int_{x-ct_0+ct}^{x+ct_0-ct} f(x,t)dxdt$$

Where $P(x_{0,}t_0)$ is any arbitrary point. From the initial conditions associated with the homogenous system, we know that

$$u_1(x,t)=\frac{1}{2}[\eta(x+ct)+\eta(x-ct)]+\frac{1}{c}\int_{x-ct}^{x+ct}\nu(\xi)d\xi$$

Hence the complete solution of the inhomogeneous wave equation in one dimensional system is given by

$$u(x,t)=u_1+u_2$$

## Two Dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{C^2}\frac{\partial^2 u}{\partial t^2} \qquad ...(1)$$

$$u(x,y,t)=X(x)\ Y(y)\,T(t)$$

Let

Be the solution of the of the above 2-D wave equation

Now

$$\frac{\partial^2 u}{\partial x^2}=YTX'',\quad \frac{\partial^2 u}{\partial y^2}=XTY'',\quad \frac{\partial^2 u}{\partial t^2}=XYT''$$

Using in (1) we get

$$YTX''+XTY''=\frac{1}{C^2}XYT''$$

Dividing throughout by $XYT$ we get

$$\frac{X''}{X}+\frac{Y''}{Y}=\frac{1}{C^2}\frac{T''}{T}$$

This will be true when each member will be a constant

Choosing the constant suitably we get

$$\frac{d^2X}{dx^2}+k^2X=0$$

And

$$\frac{d^2Y}{dy^2} + L^2X = 0$$

So that

$$\frac{d^2T}{dt^2} + \left(K^2 + L^2\right)c^2T = 0$$

The solutions of these equations are

$$X = C_1\cos Kx + C_2\sin Kx$$

$$Y = C_3\cos Ly + C_4\sin Ly$$

And

$$T = C_5\cos\left(\sqrt{K^2+L^2}\right)ct + C_6\sin\left(\sqrt{K^2+L^2}\right)ct$$

Hence the solution of two dimensional wave equation is

$$u(x, y, t = \left(C_1\cos kx + C_2\sin kx\right)\left(C_3\cos Ly + C_4\sin Ly\right)$$
$$\left(C_5\cos\left(\sqrt{K^2+L^2}\right)ct + C_6\sin\left(\sqrt{K^2+L^2}\right)ct\right)$$

**$D'$ Alembert's solution of one dimensional wave equation:**

Consider the IVP of Cauchy type described as

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}; -\infty < x < \infty, \quad t > 0 \qquad ...(1)$$

Subject to the initial conditions

$$U(x,0) = \eta(x), \qquad \frac{\partial u(x,0)}{\partial t} = v(x) \qquad ...(2)$$

Where the curves on which the initial data $\eta(x)$ and v(x) are prescribed on the $x - axis$. The functions $\eta(x)$ and v(x) are assumed to be twice continuously differentiable.

We know that the general solution of the wave equation is of the form

$$u(x, t) = f\left(x+ct\right) + g(x-ct) \qquad ...(3)$$

Where f and g are arbitrary functions

Using the given conditions

$$U(x,0) = \eta(x) = f(x) + g(x) \qquad ...(4)$$

Also,

$$\frac{\partial u(x,0)}{\partial t} = \mathrm{v(x)} = C[f'(x) - g'(x)]$$

i.e. $$C[f'(x) - g'(x)] = \mathrm{v(x)} \qquad \text{...(5)}$$

integrating (5),we get

$$f(x) - g(x) = \frac{1}{c}\int_0^x v(s)ds \qquad \text{...(6)}$$

Now adding (4)and (6),we have

$$f(x) = \frac{\eta(x)}{2} + \frac{1}{2c}\int_0^x v(s)ds$$

Also subtracting (6) from (4), we have

$$g(x) = \frac{\eta(x)}{2} - \frac{1}{2c}\int_0^x v(s)ds$$

Substituting in (3) we get

$$U(x,t) = \left(\frac{\eta(x+ct)}{2} + \frac{1}{2c}\int_0^{x+ct} v(s)ds\right) + \left(\frac{\eta(x-ct)}{2} - \frac{1}{2c}\int_0^{x-ct} v(s)ds\right)$$

$$U(x,t) = \frac{\eta(x+ct) + \eta(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} v(s)ds$$

This is known as $D'$Alembert's solution of one dimensional wave equation.

**Note:** If v=0 i.e. the string is released from rest, the solution takes the form

$$U(x,t) = \frac{\eta(x+ct) + \eta(x-ct)}{2}$$

## DUHAMELS PRINCIPLE FOR WAVE EQUATION

**STATEMENT:** Let R$^3$ be the three dimensional Euclidean space and $x = (x_1, x_2, x_3)$ be any point in R$^3$. If $v = v(x,t,\lambda)$satisfies for fixed λ the partial differential equation

$$\frac{\partial^2 v}{\partial t^2} - C^2\nabla^2 v = 0 \qquad \ldots(*)$$

with the conditions

$v(x,0,\lambda) = 0$

$$\frac{\partial v}{\partial t}(x,0,\lambda) = F(x,A) \qquad \ldots (1)$$

Where $F(x,A)$ denotes a continuous function defined for x in $R^3$

$$if u(x,t) = \int_0^t v(x,t-\lambda,\lambda) d\lambda$$

Be any continuous function, then it satisfies

$$\frac{\partial^2 v}{\partial t^2} - C^2\nabla^2 v = F(x,\lambda) \qquad \ldots (2)$$

$x \in R^3,\ t > 0$

$$u(x,0) = 0 = \frac{\partial u(x,0)}{\partial t}$$

**Proof:** We are given that V satisfies the wave equation

$$\frac{\partial^2 v}{\partial t^2} - C^2\nabla^2 v = 0$$

With the conditions given in (1)

Also for

$$u(x,t) = \int_0^t v(x,t-\lambda,\lambda) d\lambda \qquad \ldots(3)$$

To be the solution of (2) where v(x, t-λ, λ) is one parameter family solution of (*)
Also v(x,0,λ)=0 for t=λ

Differentiating eq. (3) w. r. t., $t$ under the integral sign and using Leibnitz rule we have

$$\frac{\partial u}{\partial t} = v(x,0,\lambda) + \int_0^t \frac{\partial v}{\partial t}(x,t-\lambda,\lambda) d\lambda \qquad \ldots(4)$$

Differentiating (4) again w. r. t., $t$ we have

$$\frac{\partial^2 u}{\partial t^2} = v_t(x,0,\lambda) + \int_0^t \frac{\partial^2 v}{\partial t^2}\ (x,t-\lambda,\lambda) d\lambda$$

$$= v_t(x,0,\lambda) + \int_0 c^2\nabla^2 v\ d\lambda$$

using (*)

Finally we have from eq. (3)

$$\nabla^2 u = \int_0^t \nabla^2 v$$

$$\therefore \frac{\partial^2 u}{\partial t^2} = v_t(x,0,\lambda) + C^2 \nabla^2 u$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial t^2} - C^2 \nabla^2 u = v_t(x,0,\lambda)$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial t^2} - C^2 \nabla^2 u = F(x,0,\lambda)$$

Clearly $u(x,0) = 0$ and
$\frac{\partial u(x,0)}{\partial t} = 0$

The function $v(x,t,\lambda)$ is called the pulse function or the force function.

**Exercise:** A rectangular membrane with fastened edge makes transverse vibrations. Explain how a formal series solution can be obtained.

**Sol:** The given equation in 2-D is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) 0 \le x \le a$$

$$0 \le y \le b \qquad \text{...(1)}$$

Subject to the boundary conditions

$$u(0,y,t) = u(x,0,t) = u(x,b,t) = u(a,y,t) = 0$$

And initial conditions

$$u(x,y,0) = f(x,y)$$

$$\frac{\partial u(x,y,0)}{\partial t} = g(x,y)$$

Let $u(x,y,t) = X(x)Y(y)T(t)$ be the solution of (1)

Then $\frac{\partial^2 u}{\partial x^2} = X''YT, \frac{\partial^2 u}{\partial y^2} = XY''T, \frac{\partial^2 u}{\partial z^2} = XYT''$

Substituting theses in (1) we get

$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2$ (say)

Then $T'' + \lambda^2 c^2 T = 0$

$$\Rightarrow T(t) = E\cos(\lambda ct) + F\sin(\lambda ct) \qquad \ldots(2)$$

and $\frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2$

$$\Rightarrow \frac{X''}{X} + \lambda^2 = -\frac{Y''}{Y} = \mu^2\ (say)$$

Then $\frac{X''}{X} + \lambda^2 = \mu^2$ and $\frac{Y''}{Y} = -\mu^2$

$\Rightarrow X'' + (\lambda^2 - \mu^2)X = 0$ and $Y'' + \mu^2 Y = 0$

$$\Rightarrow X = A\cos\left(\sqrt{\lambda^2-\mu^2}\,x\right) + B\sin\left(\sqrt{\lambda^2-\mu^2}x\right) \qquad \ldots(3)$$

and $Y = C\cos(\mu y) + D\sin(\mu y)$ ...(4)

put $\lambda = r$, $\sqrt{\lambda^2-\mu^2} = p$, $\mu = q$ in (2), (3) and (4) we get

$$X(x) = A\cos px + B\sin px$$

$$Y(y) = C\cos qy + D\sin qy$$

$$T(t) = E\cos rt + F\sin rt$$

Thus the solution is given by

$$u(x,y,t) = (A\cos px + B\sin px)(C\cos qy + D\sin qy)(E\cos rt + F\sin rt)$$

Now using the boundary conditions $u(0,y,t) = 0$, we get $A = 0$.

Also $u(x,0,t) = 0$, $\Rightarrow C = 0$

And $u(a,y,t) = 0 \Rightarrow \sin(pa) = 0$

$$\Rightarrow pa = m\pi \Rightarrow p = \frac{m\pi}{a}$$

Also $u(x,b,t) = 0 \Rightarrow \sin(qb) = 0$

$$\Rightarrow qb = n\pi \Rightarrow q = \frac{n\pi}{b}$$

Now using the principle of superposition we get,

$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[A_{mn}\cos(rct) + B_{mn}\sin(rct)]\left[\sin\frac{m\pi}{a}x\right]\left[\sin\frac{n\pi}{b}y\right]$ . . . (A)

where $r^2 = p^2 + q^2 = \pi^2\left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right]$

The initial condition [using in (A) ]

$$u(x,y,0) = f(x,y)$$

which implies

$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\left[\sin\frac{m\pi}{a}x\right]\left[\sin\frac{n\pi}{b}y\right]$ ... (B)

And also $\frac{\partial u(x,y,0)}{\partial t} = g(x,y)$

$g(x,y) = cr\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\left[\sin\frac{m\pi}{a}x\right]\left[\sin\frac{n\pi}{b}y\right]$ ...(C)

where

$$B_{mn} = \frac{4}{abcr}\int_0^a\int_0^b g(x,y)\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)dxdy$$

Hence (A), (B) and (C) give the required solution.

**Exercise:** Solve the IVP described by $\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial y^2} = e^x$, given that $u(x,0) = 5, \frac{\partial u(x,0)}{\partial t} = x^2$

**Sol:** Please try self.